The determinacy of infinite games specified by automata

Olivier Finkel

Equipe de Logique Mathématique
Institut de Mathématiques de Jussieu - Paris Rive Gauche
CNRS and Université Paris 7

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Some programs make a computation, get a result, and then stop. Other ones have to maintain the good behaviour of a system:

- Operating systems (Internet)
- safety systems (power plant, . . .)
- aircraft autopilot

In particular, these systems are in relation with an environment, and must have the "good" response to any changes of the environment.

INFINITE GAMES

The system in relation with an environment may be specified by an infinite game between two players.

Two players:

- Player 1 : the computer program
- Player 2 : the environment

The possible actions of the players are represented by letters of a finite alphabet A.

INFINITE PLAY

The two players compose an infinite word over the alphabet A:

Player 1 : $a_1 \quad\quad a_3 \quad\quad a_5$

$\searrow \nearrow \searrow \nearrow \searrow \quad \cdots$

Player 2 : $\quad a_2 \quad\quad a_4 \quad\quad a_6$

The infinite word $a_1.a_2.a_3\ldots$ **représents the infinite behaviour of the system.**

A good behaviour **is represented by a set of infinite words** $L \subseteq A^\omega$ **called the winning set for Player 1.**

The above game, with perfect information, is a Gale-Stewart game $G(L)$.

WINNING STRATEGIES

A strategy for Player 1 is a mapping $f : (A^2)^\star \longrightarrow A$. Player 1 follows the strategy f iff $\forall n \geq 1$: $a_{2n+1} = f(a_1 a_2 \ldots a_{2n})$.

The strategy f **is winning for Player 1 if it ensures** a good behaviour of the system, **i.e. such that :** the infinite word written by the two players belongs to the winning set L:

$$a_1.a_2.a_3 \ldots \in L$$

A winning strategy for Player 2 **is a strategy for Player 2 which ensures** that $a_1.a_2.a_3 \ldots \notin L$.

A Gale-Stewart game $G(L)$ is determined iff one of the two players has a winning strategy.

WINNING STRATEGIES

The important problems to solve in practice are:

(1) Is the game G(L) determined ?

(2) Which player has a winning strategy ?

(3) If Player 1 has a winning strategy, can we effectively construct this winning strategy ? Is it computable ?

(4) What is the complexity of this construction ? What are the necessary amounts of time and space ?

COMPLEXITY OF WINNING SETS

The winning set for Player 1 is often given as the set of infinite behaviours which satisfy a logical formula.

It is also often given as the set of infinite words accepted by a finite automaton, a one-counter automaton, a pushdown automaton, . . . with a Büchi acceptance condition . . .

Büchi acceptance condition

An automaton $\mathcal{A}$ reading infinite words over the alphabet Σ is equipped with a finite set of states K and a set of final states $F \subseteq K$.

A run of $\mathcal{A}$ reading an infinite word $\sigma \in \Sigma^\omega$ is said to be accepting iff there is some state $q_f \in F$ appearing infinitely often during the reading of σ.

An infinite word $\sigma \in \Sigma^\omega$ is accepted by $\mathcal{A}$ if there is (at least) one accepting run of $\mathcal{A}$ on σ.

An ω-language $L \subseteq \Sigma^\omega$ is accepted by $\mathcal{A}$ if it is the set of infinite words $\sigma \in \Sigma^\omega$ accepted by $\mathcal{A}$.

Context free or regular ω-languages

(Cohen and Gold 1977; Linna 1976)

Let $L \subseteq \Sigma^\omega$. Then the following propositions are equivalent :

- L is accepted by a Büchi pushdown automaton.
- $L = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$,
  for some context free finitary languages U_i and V_i.
- L is a context free ω-language.

A similar theorem holds if we:

- omit the pushdown stack and replace context free by regular,
- or replace pushdown and context-free by 1-counter.

Regular winning sets

Büchi and Landweber solved the famous Church's Problem posed in 1957, Rabin gave an alternative solution:

Theorem (Büchi-Landweber 1969; Rabin 1972)

If $L \subseteq \Sigma^\omega$ is a regular ω-language then:

- *The game $G(L)$ is determined.*
- *One can decide which Player has a winning strategy.*
- *On can construct effectively a winning strategy given by a finite state transducer.*

Deterministic context free winning sets

Walukiewicz extended this to the case of deterministic context free winning sets:

Theorem (Walukiewicz 1996)

If $L \subseteq \Sigma^\omega$ is a deterministic context free ω-language then:

- *The game $G(L)$ is determined.*
- *One can decide which Player has a winning strategy.*
- *On can construct effectively a winning strategy given by a pushdown transducer.*

Further extension to deterministic higher-order pushdown automata ([Cachat 2003], [Carayol, Hagues, Meyer, Ong, Serre 2008])

The question of the determinacy

The question remained open for non-deterministic pushdown (or even one-counter) automata.

First question: determinacy of these games ?

The determinacy of regular or deterministic context-free games follows from the determinacy of Borel games (Martin 1975).

It involves the notion of topological complexity of the winning sets. A way to study the complexity of ω-languages is to consider their topological complexity.

Topology on Σ^ω

The natural prefix metric on the set Σ^ω of ω-words over Σ is defined as follows:

For $u, v \in \Sigma^\omega$ and $u \neq v$ let

$$\delta(u, v) = 2^{-n}$$

where n is the least integer such that:

the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v.

This metric induces on Σ^ω the usual Cantor topology for which :

- *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$.
- *closed subsets* of Σ^ω are complements of *open subsets* of Σ^ω.

Borel subsets of Σ^ω

The class of Borel subsets of Σ^ω is the closure of the class of open sets

- under countable union and countable intersection, or equivalently,
- under countable union and complementation.

Complexity of ω-Languages of Non Deterministic Turing Machines

Non deterministic Büchi (or Muller) Turing machines accept **effective analytic sets** (Staiger). The class **Effective-Σ^1_1** is the class of **projections of arithmetical sets**.

There are some non-Borel sets in the class **Effective-Σ^1_1.**

Theorem

- *[Ressayre and F. 2003] There are some non-Borel context-free (and even 1-counter) ω-languages.*

The (effective) analytic determinacy

The determinacy of games $G(L)$ with L effective analytic is not provable in **ZFC**, the commonly accepted axiomatic system in which all usual mathematics can be developped.

Theorem (Martin 1970 and Harrington 1978)

The effective analytic determinacy is equivalent to the existence of a particular real called $0^\sharp$*.*
The existence of the real $0^\sharp$ *is known in set theory to be a large cardinal assumption, and is not provable in* **ZFC**.

The constructible sets in a model V of ZF

The class **L** of *constructible sets* in a model **V** of **ZF** is defined by

$$\mathbf{L} = \bigcup_{\alpha \in \mathbf{ON}} \mathbf{L}(\alpha)$$

where the sets $\mathbf{L}(\alpha)$ are constructed by induction as follows:

1. $\mathbf{L}(0) = \emptyset$
2. $\mathbf{L}(\alpha) = \bigcup_{\beta<\alpha} \mathbf{L}(\beta)$, for α a limit ordinal, and
3. $\mathbf{L}(\alpha + 1)$ is the set of subsets of $\mathbf{L}(\alpha)$ which are definable from a finite number of elements of $\mathbf{L}(\alpha)$ by a first-order formula relativized to $\mathbf{L}(\alpha)$.

If **V** is a model of **ZF** and **L** is the class of *constructible sets* of **V**, then the class **L** forms a model of **ZFC + CH.**
The axiom (**V=L**) means “every set is constructible” and is consistent with **ZFC.**

The real $0^\sharp$

A set of ordinals C is a set of indiscernibles in the constructible universe $\mathbf{L}$ iff:

• For each first-order formula $\varphi(x_1, \ldots, x_n)$ in the language of set theory,

• For all finite sequences $\alpha_{i_1} < \alpha_{i_2} < \ldots < \alpha_{i_n}$ and $\beta_{i_1} < \beta_{i_2} < \ldots < \beta_{i_n}$ of ordinals in C, it holds that:

$$\mathbf{L} \models \varphi(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}) \Longleftrightarrow \mathbf{L} \models \varphi(\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_n})$$

The real $0^\sharp$

The existence of the real $0^\sharp$ in a model **V** of **ZFC** is equivalent to the existence of an uncountable set of indiscernible ordinals in the constructible universe **L.**

(The existence of such a set was proven firstly by Silver from the existence of a Ramsey cardinal in 1966)

• The real $0^\sharp$ is the code in 2^ω of a set of integers, the set of Gödel numbers of formulas which are satisfied by an uncountable set of indiscernibles ordinals in **L.**

• The existence of the real $0^\sharp$ is equivalent to the existence of a non-trivial elementary embedding $j : \mathbf{L} \to \mathbf{L}$.

The context-free determinacy

Theorem (F. 2011)

The determinacy of games $G(L)$, where L is accepted by a real-time 1-counter Büchi automaton, is equivalent to the effective analytic determinacy and also to the existence of the real $0^\sharp$ and thus it is not provable in ZFC.

Sketch of the proof

We start from an effective analytic set $L(\mathcal{T})$ accepted by a Büchi Turing machine $\mathcal{T}$, which can be simulated by a (non real time) 2-counter automaton.

We successively construct:

- A 2-counter Büchi automaton $\mathcal{A}_1$,
- A real time 8-counter Büchi automaton $\mathcal{A}_2$,
- A real time 1-counter Büchi automaton $\mathcal{A}_3$,

such that Player 1 (resp. Player 2) has a winning strategy in $G(L(\mathcal{T}))$ if and only if Player 1 (resp. Player 2) has a winning strategy in the game $G(L(\mathcal{A}_1))$, (and similarly for $G(L(\mathcal{A}_2))$, $G(L(\mathcal{A}_3))$.

Thus the game $G(L(\mathcal{T}))$ is determined iff the game $G(L(\mathcal{A}_1))$, (resp. $G(L(\mathcal{A}_2))$, $G(L(\mathcal{A}_3))$) is determined.

Games with non-recursive strategies when they exist

Theorem (F. 2011)

There exists a 1-counter Büchi automaton $\mathcal{A}$ such that:

*(1) (**ZFC**+ $\omega_1^{\mathbf{L}} < \omega_1$): Player 1 has a winning strategy σ in the game $G(L(\mathcal{A}))$. But σ cannot be recursive and not even hyperarithmetical.*

*(2) (**ZFC** + $\omega_1^{\mathbf{L}} = \omega_1$): the game $G(L(\mathcal{A}))$ is not determined.*

Moreover these are the only two possibilities: there are no models of **ZFC** in which Player 2 has a winning strategy.

Games with non-recursive strategies when they exist

Theorem (F. 2013)

There exist a real-time 1*-counter Büchi automaton* $\mathcal{A}$ *such that the* ω*-language* $L(\mathcal{A})$ *is an arithmetical* Δ_3^0*-set and such that Player 2 has a winning strategy in the game* $G(L(\mathcal{A}))$ *but has no hyperarithmetical winning strategies in this game.*

One cannot decide who wins a 1-counter game

Theorem

There exists a recursive sequence of real time 1-counter Büchi automata $\mathcal{A}_n$, $n \geq 1$, such that all games $G(L(\mathcal{A}_n))$ are determined. But it is Π^1_2-complete (hence highly undecidable) to determine whether Player 1 has a winning strategy in the game $G(L(\mathcal{A}_n))$.

Games of maximum strength of determinacy

Theorem (F. 2012)

There exists a 1-counter Büchi automaton $A_\sharp$ such that:
The game $G(A_\sharp)$ is determined iff the effective analytic determinacy holds iff all 1-counter games are determined.

Are there two or more strengths of determinacy ?

A transfinite sequence of 1-counter Büchi automata

A transfinite sequence of games specified by real-time 1-counter Büchi automata with increasing strength of determinacy.

Theorem (F. 2012)

There is a transfinite sequence of real-time 1-counter Büchi automata $(\mathcal{A}_\alpha)_{\alpha<\omega_1^{\mathrm{CK}}}$, indexed by recursive ordinals, s.t.:

$$\forall \alpha < \beta < \omega_1^{\mathrm{CK}} \; [\; Det(G(L(\mathcal{A}_\beta))) \Longrightarrow Det(G(L(\mathcal{A}_\alpha))) \;]$$

but the converse is not true:

For each recursive ordinal α there is a model $\mathbf{V}_\alpha$ of **ZFC** *such that in this model the game $G(L(\mathcal{A}_\beta))$ is determined iff $\beta < \alpha$.*

Games specified by 2-tape Büchi automata

The two players compose an infinite word over the alphabet $A \times B$:

Player 1 : (a_1, b_1) $\searrow$ $\nearrow$ (a_3, b_3) $\searrow$ $\nearrow$ (a_5, b_5) $\searrow$ $\ldots$

Player 2 : (a_2, b_2) (a_4, b_4)

The infinite word $(a_1, b_1).(a_2, b_2).(a_3, b_3) \ldots \in (A \times B)^\omega$ **represents the infinite behaviour of the system.**

A good behaviour **is represented by a set of infinite words** $L(\mathcal{A}) \subseteq (A \times B)^\omega$ **accepted by a 2-tape Büchi automaton** $\mathcal{A}$**.**

The question of the determinacy

Theorem (F. 2012)

The determinacy of games $G(L)$, where L is accepted by a 2-tape (asynchronous) Büchi automaton, is equivalent to the effective analytic determinacy, and thus it is not provable in ZFC.

Sketch of the proof

We start from an ω-language accepted by a real time 1-counter Büchi automaton $\mathcal{A}$.

We construct, from $\mathcal{A}$, a 2-tape Büchi automaton $\mathcal{B}$ such that Player 1 (resp. Player 2) has a winning strategy in $G(L(\mathcal{A}))$ if and only if Player 1 (resp. Player 2) has a winning strategy in the game $G(L(\mathcal{B}))$.

The game $G(L(\mathcal{A}))$ is determined iff the game $G(L(\mathcal{B}))$ is determined.

Games with non-recursive strategies

Theorem (F. 2012)

There exists a 2-tape Büchi automaton $\mathcal{A}$ such that:

(1) There is a model V_1 of **ZFC** *in which Player 1 has a winning strategy σ in the game $G(L(\mathcal{A}))$. But σ cannot be recursive and not even hyperarithmetical.*

(2) There is a model V_2 of **ZFC** *in which the game $G(L(\mathcal{A}))$ is not determined.*

A transfinite sequence of 2-tape Büchi automata

A transfinite sequence of games specified by 2-tape Büchi automata with increasing strength of determinacy.

Theorem (F. 2012)

There is a transfinite sequence of 2-tape Büchi automata $(\mathcal{A}_\alpha)_{\alpha<\omega_1^{\mathrm{CK}}}$, indexed by recursive ordinals, s.t.:

$$\forall \alpha < \beta < \omega_1^{\mathrm{CK}} \ [\ \mathit{Det}(G(L(\mathcal{A}_\beta))) \Longrightarrow \mathit{Det}(G(L(\mathcal{A}_\alpha)))\ ]$$

but the converse is not true:

*For each recursive ordinal α there is a model $\mathbf{V}_\alpha$ of **ZFC** such that in this model the game $G(L(\mathcal{A}_\beta))$ is determined iff $\beta < \alpha$.*

THANK YOU !

Wadge Reducibility

Definition (Wadge 1972)

For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, $L \leq_W L'$ iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$.

L and L' are Wadge equivalent ($L \equiv_W L'$) iff $L \leq_W L'$ and $L' \leq_W L$. .

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation. The equivalence classes of $\equiv_W$ are called Wadge degrees.

Intuitively $L \leq_W L'$ means that L is less complicated than L' because to check whether $x \in L$ it suffices to check whether $f(x) \in L'$ where f is a continuous function.

Wadge Degrees

Hence the Wadge degree of an ω-language is a measure of its topological complexity.

Wadge degrees were firstly studied by Wadge for Borel sets using Wadge games.

There is a close relationship between Wadge reducibility and games:

Wadge Games

Definition (Wadge 1972)

Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. The Wadge game $W(L, L')$ is a game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'.

The two players alternatively write letters a_n of X for Player 1 and b_n of Y for player 2.
Player 2 is allowed to skip, even infinitely often, provided he really writes an ω-word in ω steps.

After ω steps, Player 1 has written an ω-word $a \in X^\omega$ and Player 2 has written $b \in Y^\omega$.
Player 2 wins the play iff $[a \in L \leftrightarrow b \in L']$, i.e. iff :

$$[(a \in L \text{ and } b \in L') \text{ or } (a \notin L \text{ and } b \notin L')].$$

Wadge Games

Theorem (Wadge)

Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. Then $L \leq_W L'$ iff Player 2 has a winning strategy in the game $W(L, L')$.

By Martin's Theorem, the Wadge game $W(L, L')$, for Borel sets L and L', is determined: One of the two players has a winning strategy.

$\longrightarrow$ Study of the Wadge hierarchy on Borel sets.

Theorem (Harrington 1978, Friedman 1971)

The determinacy of Wadge games $W(L, L')$, where L and L' are effective analytic sets, is equivalent to the determinacy of effective analytic Gale-Stewart games, and thus it is not provable in **ZFC**.

The context-free Wadge determinacy

Theorem (F. 2011)

The determinacy of Wadge games $W(L(\mathcal{A}), L(\mathcal{B}))$, where $\mathcal{A}$ and $\mathcal{B}$ are real-time 1-counter Büchi automata, is equivalent to the effective analytic (Wadge) determinacy, and thus it is not provable in **ZFC**.

The Topological complexity of a 1-counter ω-language depends on the models of ZFC

Theorem (F. 2009)

There exists a 1-counter Büchi automaton $\mathcal{A}$ such that the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system **ZFC**.

1. *There is a model V_1 of* **ZFC** *in which the ω-language $L(\mathcal{A})$ is an analytic but non Borel set.*
2. *There is a model V_2 of* **ZFC** *in which the ω-language $L(\mathcal{A})$ is a G_δ-set (i.e. Π^0_2-set).*

Wadge Games Between 1-Counter Automata

The ω-language $(0^\star \cdot 1)^\omega \subseteq \{0,1\}^\omega$ is ω-regular, accepted by a Büchi automaton $\mathcal{B}$, and is Π^0_2-complete in every model of **ZFC**. This implies:

Theorem (F. 2010)

*There exists a 1-counter Büchi automaton $\mathcal{A}$ and a Büchi automaton $\mathcal{B}$ such that $L(\mathcal{A}) \leq_W L(\mathcal{B})$ is independent from **ZFC**:*

*(1) There is a model V_1 of **ZFC** in which Player 2 has a winning strategy σ in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$. But σ is not recursive and not even hyperarithmetical.*

*(2) There is a model V_2 of **ZFC** in which Player 2 has no winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$. Moreover Player 1 has no winning strategy, and the game is not determined.*