The Complexity of Maximal Block Functions of η-like Computable Linear Oderings.

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Outline

Section Guide

Introduction.

The study of the algorithmic combinatorics within which the present talk can be viewed involves the study of structures that are presented in some algorithmic manner. Loosely speaking we are given some algorithmic presentation of a structure and we ask whether we can derive algorithms for various features of the structure. A classical instance of this is the word problem for groups where we are give a “finite presentation”

$$G = \langle x_1, \ldots, x_n : y_1, \ldots, y_m \rangle$$

and one asks if there is an algorithm to decide if a word in $x_1, \ldots, x_n$ is the identity or not in G. The well known result by Novikov and Boone shows that one can pick G so that the answer is no.

Typical Questions 1.

The effective content of classical theorems. For example, we know that every infinite linear ordering has an infinite ω or $\omega*$ sequence.

- Is this true effectively: does every infinite computable linear ordering have an effective ω or ω^* sequence? Answer: no (Tennenbaum, Dennisov).
- An effective version? Rosenstein showed that if $\mathscr{L}$ is a computable linear ordering then $\mathscr{L}$ has a computable sequence of order type ω, ω^*, $\omega + \omega^*$, or $\omega + \zeta \cdot \eta + \omega^*$.
- Lerman proved that all these types are necessary.

Typical Questions 2.

The complexity of suborderings/ Effective rigidity properties.

- If τ is the condensation type of a computable linear ordering, what is the arithmetical complexity of τ?
  - Watnick showed (for example) that τ is an order type with a Π^0_2 copy iff $\zeta \cdot \tau$ has a computable copy.
- Conjecture (Kierstead): every computable copy of a computable linear order type τ has a computable self embedding iff τ has a strongly η-like interval.
  - Downey, Kastermans, Lempp (2009): true if τ is η-like.
  - Moses (2011) extended this result: true if τ has condensation type η.

Section Guide

Background Notation

- We assume $\{W_e\}_{e\in\mathbb{N}}$ to be a standard listing of c.e. sets with associated c.e. approximations $\{W_{e,s}\}_{e,s\in\mathbb{N}}$. $\emptyset'$ denotes the standard halting set for Turing machines in this context, i.e. the set $\{\, e \mid e \in W_e \,\}$ and $\mathbf{0'}$ denotes the Turing degree of $\emptyset'$.
- We suppose $q_0, q_1, q_2, \ldots$ to be a fixed computable listing of $\mathbb{Q}$.
- We also assume $\langle x, y\rangle$ to be a standard computable pairing function over $\mathbb{N}$ extended to use over $\mathbb{Q}$ via the above listing.
- $\{D_n\}_{n\in\mathbb{N}}$ denotes the canonical computable listing of all finite sets of nonnegative integers. Note that under this listing, for any $m, n \in \mathbb{N}$, if $D_m \subseteq D_n$ then $m \leq n$.
- For any function F with domain and range in $\mathbb{N}$ or $\mathbb{Q}$ we use $G(F)$ to denote the set $\{\, \langle x, y\rangle \mid F(x)\downarrow = y \,\}$, i.e. the graph of F coded into $\mathbb{N}$ via the pairing function $\langle \cdot, \cdot \rangle$.
- We define F to be Γ, for some predicate of sets Γ, if $G(F) \in \Gamma$.

Arithmetical Complexity Results

- $U_{e,s}$ is shorthand for the (finite) set defined by some fixed universal computable characteristic function $\lambda n\, U(e,n,s)$.
- There is $\{U_{e,s}\}_{e,s\in\mathbb{N}}$ such that a set A is Σ^0_2 iff for some e,

$$A = \{\, n \mid \exists t(\forall s \geq t)[\, n \in U_{e,s}\,]\,\}\,.$$

- There is $\{U_{e,s}\}_{e,s\in\mathbb{N}}$ such that a set B is Π^0_2 iff for some e,

$$B = \{\, n \mid \forall t(\exists s \geq t)[\, n \in U_{e,s}\,]\,\}\,.$$

- A set C is Δ^0_2 iff there exists computable $\lambda n U(n,s)$ such that, $\lim_{s\to\infty} U(n,s)$ exists for all n and $\lim_{s\to\infty} U(n,s) = 1$.
- A set D is Δ^0_{n+1} iff D is computable in $\emptyset^n$ (i.e. $D \leq_{\mathrm{T}} \emptyset^n$).

Linear Orders: Notation/Results

- We use η to denote the order type of $\mathbb{Q}$ whereas n denotes the finite order type with n elements. For linear orders $\mathscr{L}_\beta = \langle L_\beta, <_{\mathscr{L}_\beta} \rangle$ and $\mathscr{L}_\gamma = \langle L_\gamma, <_{\mathscr{L}_\gamma} \rangle$ of order type β and γ respectively, $\beta \cdot \gamma$ denotes the order type of $\mathscr{L}_\beta \times \mathscr{L}_\gamma$ under lexicographical ordering (from the right). For example $2 \cdot \eta$ denotes the order type of a linear ordering formed by taking a copy of the rational numbers and replacing every element by an ordered pair.
- Let $\mathscr{L} = \langle L, <_{\mathscr{L}} \rangle$ be a linear ordering. We call $S \subseteq L$ an *interval* if, for all $a, b \in S$, and any c that lies $<_{\mathscr{L}}$ between a and b, c is also in S. For any $a, b \in L$, we say that a, b are *finitely far apart*—written $B_{\mathscr{L}}(a, b)$—if the interval S of elements lying between a and b is finite.

Linear Orders: Notation/Results

- Noting that $B_{\mathscr{L}}$ is an equivalence relation we say that the *condensation type* of $\mathscr{L}$ is the order type of the quotient of $\mathscr{L}$ by $B_{\mathscr{L}}$. Note also that we call $B_{\mathscr{L}}$ the *block relation* of $\mathscr{L}$.
- If $\mathscr{L}$ is countably infinite we define $\mathscr{L}$ to be η*-like* if $\{\, c \mid B_{\mathscr{L}}(c, a) \,\}$ is finite for all $a \in L$ or, equivalently, if $\mathscr{L}$ has order type $\sum\{\, F(q) \mid q \in \mathbb{Q} \,\}$ for some function $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$.
- We call any finite interval in $\mathscr{L}$ a *block* and we call the equivalence classes under $B_{\mathscr{L}}$ *maximal* blocks. If $\mathscr{L}$ is η-like we call any such associated function F a *maximal block function* of $\mathscr{L}$.

Linear Orders: Notation/Results

- We say that $\mathscr{L}$ is *strongly η-like* if in addition F has finite range (i.e. the maximal block size is bounded).
- For any distinct elements $a, b \in L$ we say that a and b are *adjacent*—written $N_{\mathscr{L}}(a, b)$—if the interval of elements lying between a and b is empty. Note that $\neg N_{\mathscr{L}}$ is computably enumerable in $<_{\mathscr{L}}$.
- If $\mathscr{L} = \langle L, <_{\mathscr{L}} \rangle$ is countably infinite we derive a listing $l_0, l_1, l_2, \ldots$ of L computable in $<_{\mathscr{L}}$. This allows us to assume that $L = \mathbb{N}$. We say that $\mathscr{A}$ is *computable* if $<_{\mathscr{L}}$ is computable.
- If $\mathscr{L} = \langle L, <_{\mathscr{L}} \rangle$ is computable then (by the above) $N_{\mathscr{L}}$ is Π^0_1 and so $B_{\mathscr{L}}$ is Σ^0_2.

New Notation/Results

- A function $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ is infinum Π^0_n if, for some Π^0_n set U_j, $F(q_e) = \mu n [\, \langle q_e, n \rangle \in U_j \,]$.
- Note that F is infinum Π^0_{n+1} iff F is $\mathbf{0}^n$ limitwise monotonic (Khisamiev).
- Note also that F is infinum Π^0_2 iff $F(q_e) = \liminf_{s \to \infty} \widehat{F}(q_e, s)$ for some computable function $\widehat{F}$.

New Notation/Results

Lemma (H 2014)

If $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ is infinum Π^0_2, then $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q}\,\}$ has a computable presentation.

Proof.

Given Π^0_2 U_j, construct $\mathscr{L} = \langle L, <_{\mathscr{L}} \rangle$ with $L = \mathbb{N}$ by stages with finite $\mathscr{L}_s = \langle L_s, <_{\mathscr{L}} \rangle$ defined at the end of each stage. Using $q_0, q_1, q_2, \ldots$ define the approximation $I(e, s)$ to block $I(e)$ for all $e \leq s$: $|I(e, s)| = \widehat{F}(q_e, s) =_{\text{def}} \mu n [\, \langle q_e, n \rangle \in U_{j,s} \cup \{s\}\,]$ so that $F(q_e) = \mu n[\, \langle q_e, n \rangle \in U_i\,] = |I(e)|$. □

Section Guide

Background Results

Theorem (Fellner 1976)

If $\mathscr{B}$ is a computable η-like linear ordering then there is a Δ^0_3 function F such that $\mathscr{B}$ has order type $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q} \,\}$.

Lemma (Jockush 1968)

There exists a computable listing $\{U_e\}_{e\in\mathbb{N}}$ of the Π^0_2 sets with associated computable approximation $\{U_{e,s}\}_{e,s\in\mathbb{N}}$ satisfying, for all $e \geq 0$, $U_e = \{\, x \mid \forall t(\exists s \geq t)[\, x \in U_{e,s} \,] \,\}$ and such that, for any finite sets $E_0, \ldots, E_e$ with $E_i \subseteq U_i$ for all $0 \leq i \leq e$, there exist infinitely many stages s such that $E_i \subseteq U_{i,s}$ for all $0 \leq i \leq e$.

Main Theorem

Theorem (H 2014)

There exists a computable linear ordering $\mathscr{L}$ of order type $\kappa = \sum\{\, F(q) \mid q \in \mathbb{Q}\,\}$ such that $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$, and such that, for any Π^0_2 function $G : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ and linear ordering $\mathscr{B} \cong \mathscr{L}$, $\mathscr{B}$ does not have order type $\tau = \sum\{\, G(q) \mid q \in \mathbb{Q}\,\}$.

The Requirements.

The construction aims to construct $\mathscr{L}$ of order type $\sum\{\, F(q) \mid q \in \mathbb{Q}\,\}$ such that $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ and such that F satisfies, for all $e \in \mathbb{N}$, the following requirements:

R_e : for all $k, j \leq e$, either

1. $\langle q_k, F(q_e)\rangle \notin U_j$, or
2. there exist $m \neq l$ such that $\langle q_k, m\rangle \in U_j$ and $\langle q_k, l\rangle \in U_j$.

Elements of the Construction.

Our Aim. Construct a computable linear ordering $\mathscr{L} = \langle L, <_{\mathscr{L}} \rangle$ with domain $L = \mathbb{N}$ arranged in a set of maximal blocks $\{ I(n) \mid n \in \mathbb{N} \}$ such that, for all $n \geq 0$, $F(q_n) = |I(n)|$ and also such that $I(n)$ is ordered relative to $\{ I(k) \mid k \neq n \}$ as q_n is ordered relative to $\{ q_k \mid k \neq n \}$; i.e. under our present terminology, such that the listing $I(0), I(1), I(2), \ldots$ is an assignment of F to $\mathscr{L}$.

Requirements? Satisfaction of $\{R_e\}_{e \in \mathbb{N}}$ ensures that, for any $j \in \mathbb{N}$, if U_j is the graph of a maximal block function G_j and $\mathscr{B}$ is a linear ordering of order type $\gamma = \sum\{ G_j(q) \mid q \in \mathbb{Q} \}$, then $\mathscr{B} \not\cong \mathscr{L}$.

The Diagonalisation Witness and Domain.

Diagonalisation Witness. For $s \geq e$, witness $m(e,s)$ for R_e is the construction's guess as to a number such that $\langle q_k, m(e,s)\rangle \notin U_j$, for all $0 \leq k, j \leq e$ such that U_j is the graph of a function $\mathbb{Q} \to \mathbb{N} \setminus \{0\}$. For all stages $s > e$, $m(e,s) = |I(e,s)| = F_s(e)$, where F_s is the s stage approximation to F.

The set of diagonalisation pairs. For index e define

$$P^e \;=\; \{\, (i,j) \mid 0 \leq i,j \leq e \,\}\,.$$

Thus $|P^e| = (1+e)^2$. Letting $x_0^e, \ldots, x_{(1+e)^2-1}^e$ be the computable ordering of P^e induced by the standard pairing function $\langle \cdot, \cdot \rangle$ we have the computable listing $D_0^e, \ldots, D_{2^{(1+e)^2}-1}^e$ of all subsets of P^e. (Note: $D_i^e \subseteq D_j^e \Rightarrow i \leq j$.)

The Diagonalisation Witness and Domain.

The diagonalisation domain. For R_e define this to be:

$$Z_e = X_0^e \cup \ldots \cup X^e_{2^{(1+e)^2}-1}$$

where, for each $0 \leq i \leq 2^{(1+e)^2} - 1$, X_i^e is an interval of numbers associated with D_i^e such that (i) $|X_i^e| \geq |D_i^e| + 1$ and (ii) for $i \neq 0$, $\min X_i^e > \max X_{i-1}^e$. To do this, for simplicity we define X_i^e such that $|X_i^e| = (1+e)^2 + 1$ for every $0 \leq i \leq 2^{(1+e)^2} - 1$ by defining:

$$X_i^e = \{i(1+e)^2 + (i+1), \ldots, (i+1)(1+e)^2 + (i+1)\}.$$

Accordingly Z_e is the interval $\{1, \ldots, |Z_e|\}$ partitioned by the X_i^e and having cardinality $|Z_e| = 2^{(1+e)^2}((1+e)^2 + 1)$.

The Diagonalisation.

Approximating $F(q_e) = m(e)$. The point here is that, at stage s we choose an index $i(e,s)$ such that:

$$(k,j) \in D^e_{i(e,s)} \Leftrightarrow |\{ \langle q_k, r\rangle \mid r \in Z_e \} \cap U_{j,s}| \leq 1$$

for all $0 \leq k, j \leq e$. Since $|X^e_{i(e,s)}| \geq |D_{i(e,s)}| + 1$ we know that

$$X_{i(e,s)} \setminus \{ r \mid r \in Z_e \ \& \ (\exists (k,j) \in D_{i(e,s)})[\langle q_k, r\rangle \in U_{j,s}] \} \neq \emptyset$$

so that we can define the s stage witness $m(e,s)$ to be a number in this set.

Verification 1.

Definition of $i(e)$. Let $i(e)$ be the index satisfying

$$(k,j) \in D^e_{i(e)} \Leftrightarrow |\{\, \langle q_k, r\rangle \mid r \in Z_e \,\} \cap U_j| \leq 1$$

for all $0 \leq k,j \leq e$.

Result 1. Let $t_e > s_e$ be a stage such that

$$|\{\, \langle q_k, r\rangle \mid r \in Z_e \,\} \cap U_{j,s}| \;\leq\; 1$$

for all $(k,j) \in D^e_{i(e)}$ and $s \geq t_e$. Then, by definition, at any such stage s, $D^e_{i(e)} \subseteq D^e_{i(e,s)}$ and so $i(e) \leq i(e,s)$.

Verification 2.

Result 2. For each $0 \leq j \leq e$ define

$$E_j = \{ \langle q_k, r \rangle \mid r \in Z_e \ \& \ k \leq e \ \& \ (k,j) \notin D^e_{i(e)} \} \cap U_j$$

By the earlier Lemma, there are infinitely many stages s such that $E_j \subseteq U_{j,s}$ for all $0 \leq j \leq e$. Moreover, at each such stage $s \geq t_e$, by definition of the construction, $i(e) = i(e,s)$.

Result 3. Hence we can define $F(q_e) = m(e)$ where $m(e) \in X_{i(e)}$, so that $m(e) \leq m(e,s)$ for all $s \geq t_e$ and $m(e) = m(e,s)$ for ∞ many stages s. Note that $F(q_e)$ satisfies R_e.

Remark. $F(q_e) = \mu m[\langle q_e, m \rangle \in U]$ where

$$U = \{ \langle q_e, m \rangle \mid \forall t (\exists s \geq t)[m(e,s) = m] \} .$$

Verification 3.

Note. Suppose that $\mathscr{B}$ is a linear ordering and $\iota : \mathscr{B} \cong \mathscr{L}$ is an isomorphism. Suppose also that $\widehat{F} : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ is a maximal block function of $\mathscr{B}$ and that $\widehat{I}(0), \widehat{I}(1), \widehat{I}(2), \ldots$ is an assignment of $\widehat{F}$ to $\mathscr{B}$. Now note that we have a listing of labels of $\mathscr{L}$,

$$m_0, m_1, m_2, \ldots$$

such that

$$\iota : \widehat{I}(j) \;\cong\; I(m_j)$$

(i.e. $I(m_j)$ is the isomorphic image of $\widehat{I}(j)$ under ι) for all $j \geq 0$. Moreover there must be infinitely many labels j of $\mathscr{B}$ such that $m_j \geq j$. We therefore conclude that there are infinitely many pairs of labels (k, e) with $k \leq e$ such that $\iota : \widehat{I}(k) \cong I(e)$.

Verification 4.

Result 4. Choose any $\mathscr{B}$, ι, $\widehat{F}$ and assignment $\widehat{I}(0), \widehat{I}(1), \widehat{I}(2), \ldots$ as in the Note. Consider any index $j \geq 0$ and suppose that U_j is the graph of a function G_j with domain $\mathbb{Q}$. As above, choose $k \geq j$ such that $\iota : \widehat{I}(k) \cong I(e)$ for some $e \geq k$. Now, by definition of the construction, $(k, j) \in D_{i(e)}$. However this implies that

$$G_j(q_k) \;\neq\; m(e) \;=\; F(q_e) \;=\; |I(e)| \;=\; |\widehat{I}(k)|\,.$$

Note that the choice of $\widehat{F}$, and of its assignment to $\mathscr{B}$, as also of the isomorphism $\iota : \mathscr{B} \cong \mathscr{L}$, was in each case arbitrary. The same holds for the choice of the index $j \geq 0$, and of the linear ordering $\mathscr{B} \cong \mathscr{L}$. We can thus conclude that, for *any* Π^0_2 function $G : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ and *any* $\mathscr{B} \cong \mathscr{L}$, $\mathscr{B}$ does not have order type $\tau = \sum\{\, G(q) \mid q \in \mathbb{Q}\,\}$.

Section Guide

Further Results.

Lemma (H 2014)

If computable $\mathscr{L} = \langle L, <_{\mathscr{L}} \rangle$ is strongly η-like or if $\mathscr{L}$ is η-like but contains no strongly η-like interval then $\mathscr{L}$ has order type $\sum\{ F(q) \mid q \in \mathbb{Q} \}$ for some infinum Π^0_2 function F.

Further Results.

Conjecture (Kierstead 1987). Every computable presentation of a computable linear order type τ has a strongly nontrivial Π^0_1 automorphism if and only if τ contains an interval of order type η.

Lemma (Kierstead 1987)

There exists a computable linear order $\mathscr{B}$ of order type $2 \cdot \eta$ that is Π^0_1-rigid.

Note. It is easy to construct a computable nontrivial automorphism of any computable $\mathscr{L}$ if $\mathscr{L}$ has an interval of order type η. Similarly, if $\mathscr{L}$ has an interval of order type $n \cdot \eta$ for some $n > 1$ then $\mathscr{L}$ has a Δ^0_2 nontrivial automorphism.

Further Results.

Theorem (H, Lee and Cooper 2014)

Suppose that $\mathscr{B}$ is an η-like computable linear ordering with no interval of order type η, such that $\mathscr{B}$ has order type $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q}\,\}$ for some infinum Π^0_2 function F. Then for any graph subuniform Δ^0_2 class $\mathcal{F}$ there is a computable $\mathscr{L} \cong \mathscr{B}$ which is $\mathcal{F}$-rigid.

Note. Examples of *graph subuniform* Δ^0_2 classes.

(i) The class of Π^0_1 functions. (ii) The class of α-c.e. functions, for any ordinal $\alpha < \omega^2$. (iii) The class of functions (whose graphs are) a-c.e. for some $a \in \mathcal{A}$ where $\mathcal{A}$ is a Σ^0_2 subset of the set of Kleene notations $\mathcal{O}$ for the computable ordinals. (iv) The class of functions computable in a set B if B is *low*.

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Background References.

Good sources for Computable Linear Orderings are Downey's survey paper and Rosenstein's monograph as detailed below.

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