On Maximal Block Functions of Computable η-like Linear Orderings.

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Abstract. We prove the existence of a computable η-like linear ordering $\mathscr{L}$ such that, for any Π^0_2 function $G : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ and linear ordering $\mathscr{B} \cong \mathscr{L}$, $\mathscr{B}$ does not have order type $\tau = \sum\{\, G(q) \mid q \in \mathbb{Q}\,\}$.

Keywords: Computable, linear ordering, η-like, Π^0_2, maximal block.

1 Introduction.

Given their relative simplicity, the study of η-like linear orderings has attracted attention as a preliminary test case for obtaining general results for computable linear orderings. An example of this is Kierstead's [Kie87] construction of a computable linear ordering of order type $2 \cdot \eta$ with no nontrivial Π^0_1 automorphism, and subsequent conjecture that every computable copy of a computable linear ordering $\mathscr{L}$ has a strongly nontrivial[1] Π^0_1 automorphism if and only if the order type τ of $\mathscr{L}$ contains an interval of order type η. This conjecture is supported by the Theorem in [DM89] that every computable discrete linear ordering $\mathscr{L}$ has a computable copy with no strongly nontrivial Π^0_1 self embedding. In the context of η-like linear orderings, Downey and Moses deduced that Kierstead's result for the order type $2 \cdot \eta$ can be generalised to the case of any η-like order type[2] τ provided that τ has a Π^0_2 maximal block function and no interval of order type

[1] Kierstead defines an automorphism f of a linear ordering $\mathscr{L}$ to be *fairly trivial* if it is nontrivial but maps every element x to an element y with $[x, y]$ finite and f to be *strongly nontrivial* if it is neither trivial nor fairly nontrivial. Note that if $\mathscr{L}$ is η-like then any nontrivial automorphism of $\mathscr{L}$ is strongly nontrivial.

[2] The powerful *choice set* method—a choice set of a linear ordering is a set containing precisely one element from each maximal block—used by Moses and Downey in the context of embeddings of discrete linear orderings [DM89], would need to be modified in order to be applicable to *any* such order type τ. Indeed suppose that $\mathscr{L}$ is a computable η-like linear ordering with a strongly η-like interval. Choose elements $a <_{\mathscr{L}} b$ in one such interval such that a is the rightmost and b the leftmost element of its respective maximal block, and such that, for some n, the interval (a, b) contains infinitely many maximal blocks of size n and no maximal block of size $m > n$. Then the set of leftmost elements (and, in fact, for any $1 \leq i \leq n$, the set of *i to leftmost* elements) of the maximal blocks of size n in the interval (a, b) forms an infinite Σ^0_2 set. Hence any construction that diagonalises against Σ^0_2 subsets of a

η. This last result is the starting point of the present paper as it prompts the question of whether it can be applied to the whole class of computable η-like linear orderings and hence, in particular, of whether every computable η-like linear ordering $\mathscr{L}$ has a copy $\mathscr{B}$ with a Π^0_2 maximal block function. We answer this question in the negative in Theorem 2 by constructing a counterexample $\mathscr{B}$ via a diagonalisation argument applied using the basic properties of isomorphisms of linear orderings in this context. We note that this solves a question mentioned by several authors including Fellner [Fel76], Lerman and Rosenstein [LR82] and Downey and Moses [DM89].

2 Preliminaries.

We assume $\{W_e\}_{e\in\mathbb{N}}$ to be a standard listing of c.e. sets with associated c.e. approximation $\{W_{e,s}\}_{e,s\in\mathbb{N}}$. $\emptyset'$ denotes the standard halting set for Turing machines in this context, i.e. the set $\{\, e \mid e \in W_e \,\}$ and $\mathbf{0'}$ denotes the Turing degree of $\emptyset'$. We suppose $q_0, q_1, q_2, \ldots$ to be a fixed computable listing of $\mathbb{Q}$. We also assume $\langle x, y\rangle$ to be a standard computable pairing function over $\mathbb{N}$ extended to use over $\mathbb{Q}$ via the above listing. $\{D_n\}_{n\in\mathbb{N}}$ denotes the canonical computable listing of all finite sets of nonnegative integers. Note that under this listing, for any $m, n \in \mathbb{N}$, if $D_m \subseteq D_n$ then $m \leq n$.

For any set X, we use $|X|$ to denote the cardinality of X. For any function[3] F with domain and range in $\mathbb{N}$ or $\mathbb{Q}$ we use $G(F)$ to denote the set $\{\, \langle x, y\rangle \mid F(x){\downarrow} = y \,\}$, i.e. the graph of F coded into $\mathbb{N}$ via the pairing function $\langle \cdot, \cdot \rangle$. (Note that in this context we identify a pair (x, y) with its code $\langle x, y\rangle$ so that, for example, the shorthand $G(F) \subseteq \mathbb{Q} \times \mathbb{N}$ makes sense.) We define F to be Γ, for some predicate of sets Γ, if $G(F) \in \Gamma$.

Note 1. Any[4] Σ^0_2 function F with domain $\mathbb{Q}$ and codomain $\mathbb{N}$ is Δ^0_2. Indeed using a $\mathbf{0'}$ oracle we can compute $F(q)$, for any $q \in \mathbb{Q}$, as the number n found by enumerating $G(F)$ until we find $\langle x, n\rangle$ with $x = q$.

Let $\mathscr{L} = \langle L, <_{\mathscr{L}}\rangle$ be a linear ordering. We call $S \subseteq L$ an *interval* if, for all $a, b \in S$, and any c that lies $<_{\mathscr{L}}$ between a and b, c is also in S. Notice that S does not necessarily have endpoints. Note that we also use the term *interval* in direct reference to the order type of $\mathscr{L}$ with obvious meaning. For any $a, b \in L$, we say that a, b are *finitely far apart*—written $a \sim^* b$—if the interval S of elements lying between a and b is finite. (By definition $S = \emptyset$ if $a = b$.) Note

choice set containing, for example, the leftmost element in each maximal block, will not be applicable if $\mathscr{L}$ contains a strongly η-like interval. (Note that a proof based on similar techniques to those used in [Kie87] can be applied in this context—see [HLC14].)

[3] We use the convention here and in further work that maximal block functions are usually denoted using upper case letters whereas automorphisms of linear orderings are usually denoted using lower case letters.

[4] This is a particular case of the same (standard) observation generalised from $n = 2$ to any $n \geq 1$, when the domain (and codomain) of F are computable.

that $\sim^*$ is an equivalence relation. If $\mathscr{L}$ is countably infinite we define $\mathscr{L}$ to be *η-like* if $\{\, c \mid c \sim^* a \,\}$ is finite for all $a \in L$ or, equivalently, if $\mathscr{L}$ has order type $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q} \,\}$ for some function $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$. We call any finite interval in $\mathscr{L}$ a *block* and we call the equivalence classes under $\sim^*$ *maximal* blocks. If $\mathscr{L}$ is η-like we call such a function F a *maximal block function* of $\mathscr{L}$. We say that $\mathscr{L}$ is *strongly η-like* if in addition F has finite range (i.e. the maximal block size is bounded). For any maximal block I of size $p \geq 1$ (written $|I| = p$) we use terminology of the form $I = k_1 <_{\mathscr{L}} \cdots <_{\mathscr{L}} k_p$ to denote I and we call k_1 (k_p) the *leftmost* (*rightmost*) element of I. If $\mathscr{A} = \langle A, <_{\mathscr{A}} \rangle$ is a countably infinite linear ordering we assume that $A = \mathbb{N}$ and derive a listing $a_0, a_1, a_2, \ldots$ of A computable in $<_{\mathscr{A}}$. We say that $\mathscr{A}$ is *computable* if $<_{\mathscr{A}}$ is computable.

We assume the reader to be conversant with the Arithmetical Hiearchy and Turing reducibility ($\leq_{\mathrm{T}}$). We refer the reader to [Odi89] for further background and notation in computability theory and to [Dow98] for a review of computability thoeretic results in the context of linear orders.

3 The Complexity of Maximal Block Functions.

Fellner determined a bound for the arithmetical complexity of maximal block functions of a computable η-like linear ordering.

Theorem 1 ([Fel76]). *If $\mathscr{B}$ is a computable η-like linear ordering then there is a Δ^0_3 function F such that $\mathscr{B}$ has order type $\tau = \sum\{\, F(q) \mid q \in \mathbb{Q} \,\}$.*

Our present concern is with the extent to which the bound in Theorem 1 can be tightened. However before proceeding we need to take into account that care is needed when dealing with the notion of maximal block functions for η-like linear orderings.

Note 2. Let $\mathscr{A}$ be an η-like linear ordering. Then $\mathscr{A}$ may have many different maximal block functions. For example, if $\mathscr{A}$ contains maximal blocks of size $n+1$ for all $n \geq 0$ then, for each $n \geq 0$ we can define a distinct maximal block function F_n for $\mathscr{A}$ such that $F_n(q_0) = n+1$.

Note 3. If $\mathscr{A}$ is an η-like linear ordering and F is a maximal block function of $\mathscr{A}$ we say that a listing $I(0), I(1), I(2), \ldots$ of maximal blocks of $\mathscr{A}$ is an *assignment* of F to $\mathscr{A}$ if $F(q_n) = |I(n)|$ for all $n \geq 0$. Note that there may be many different such assignments of F to $\mathscr{A}$. For example[5], suppose that $\mathscr{A}$ is made up of sets of maximal blocks of size 2 and 3 and that each of these sets is dense (in the standard sense) in $\mathscr{A}$. Let $I(0), I(1), I(2), \ldots$ be some listing of the maximal blocks of $\mathscr{A}$ and let $I(i_0), I(i_1), I(i_2), \ldots$ be a sublisting of maximal blocks of size 2. Then we can define a block function F of $\mathscr{A}$ with $F(q_0) = 2$ such that for every $k \geq 0$ there is a distinct assignment $I_k(0), I_k(1), I_k(2), \ldots$ of F to $\mathscr{A}$ such that $I_k(0) = I(i_k)$.

[5] An even easier but less interesting example of this phenomemon is when $\mathscr{A}$ has order type $n \cdot \eta$ for some $n \geq 1$.

Our next Lemma restates a well known property of the class of Π^0_2 sets, originally proved by Jockusch [Joc68], in a form directly applicable to our main Theorem.

Lemma 1. *There exists a computable listing* $\{U_e\}_{e\in\mathbb{N}}$ *of the* Π^0_2 *sets with associated computable approximation* $\{U_{e,s}\}_{e,s\in\mathbb{N}}$ *satisfying, for all* $e \geq 0$, $U_e = \{\, x \mid \forall t(\exists s \geq t)[\, x \in U_{e,s}\,]\,\}$ *and such that, for any finite sets* $E_0, \ldots, E_e$ *with* $E_i \subseteq U_i$ *for all* $0 \leq i \leq e$*, there exist infinitely many stages* s *such that* $E_i \subseteq U_{i,s}$ *for all* $0 \leq i \leq e$.

We now proceed to our main Theorem.

Theorem 2. *There exists a computable linear ordering* $\mathscr{L}$ *of order type* $\kappa = \sum\{\, F(q) \mid q \in \mathbb{Q}\,\}$ *such that* $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$*, and such that, for any* Π^0_2 *function* $G : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ *and linear ordering* $\mathscr{B} \cong \mathscr{L}$, $\mathscr{B}$ *does not have order type* $\tau = \sum\{\, G(q) \mid q \in \mathbb{Q}\,\}$.

Note. By Note 1 we can replace "Π^0_2" by "$\Sigma^0_2 \cup \Pi^0_2$" in the statement of Theorem 2. Notice that, taken in conjunction with Theorem 1, this implies that any *computable* $\mathscr{B} \cong \mathscr{L}$ has a properly Δ^0_3 maximal block function. In particular we will see that this is the case for the function F constructed below.

Proof. Assume $\{U_e\}_{e\in\mathbb{N}}$ to be a standard listing of the class of Π^0_2 sets with associated computable Π^0_2 approximation $\{U_{e,s}\}_{e,s\in\mathbb{N}}$ as prescribed by Lemma 1. The construction aims to construct $\mathscr{L}$ of order type $\sum\{\, F(q) \mid q \in \mathbb{Q}\,\}$ such that $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ and such that F satisfies, for all $e \in \mathbb{N}$, the following requirements:

$$R_e : (\forall k, j \leq e)[\, \langle q_k, F(e)\rangle \notin U_j \;\vee\; \exists m \exists l[\, m \neq l \;\&\; \langle q_k, m\rangle \in U_j \;\&\; \langle q_k, l\rangle \in U_j\,]\,].$$

We shall see in the course of the verification below that satisfaction of $\{R_e\}_{e\in\mathbb{N}}$ ensures that, for any $j \in \mathbb{N}$, if U_j is the graph of a maximal block function G_j and $\mathscr{B}$ is a linear ordering of order type $\gamma = \sum\{\, G_j(q) \mid q \in \mathbb{Q}\,\}$, then $\mathscr{B} \ncong \mathscr{L}$.

For clarity, we use $<_{\mathbb{Q}}$ and $<_{\mathbb{N}}$ when we need to differentiate between the respective standard orderings of $\mathbb{Q}$ and $\mathbb{N}$. Our aim is to construct a computable linear ordering $\mathscr{L} = \langle L, <_{\mathscr{L}}\rangle$ with domain $L = \mathbb{N}$ arranged in a set of maximal blocks $\{\, I(n) \mid n \in \mathbb{N}\,\}$ such that, for all $n \geq 0$, $F(q_n) = |I(n)|$ and also such that $I(n)$ is ordered relative to $\{\, I(k) \mid k \neq n\,\}$ as q_n is ordered relative to $\{\, q_k \mid k \neq n\,\}$; i.e. under our present terminology, such that the listing $I(0), I(1), I(2), \ldots$ is an assignment of F to $\mathscr{L}$.

We will proceed by stages $s \geq 0$ defining a finite linear ordering $\mathscr{L}_s = \langle L_s, <^s_{\mathscr{L}}\rangle$ at stage s such that, for some $n_s, r_s \geq 0$, $L_s = \mathbb{N}\!\upharpoonright\! n_s$ and such that $\mathscr{L}_s$ is arranged as a finite set of blocks $\{\, I(n, s) \mid n < r_s\,\}$ where, for all $n < r_s$, $I(n, s)$ is the s stage approximation to maximal block $I(n)$. We say that n is the *label* of $I(n, s)$ and use this terminology quite generally in order to distinguish this use of $\mathbb{N}$ from our use of $\mathbb{N}$ as the domain of the linear ordering. The ordering $<^s_{\mathscr{L}}$ is defined by the internal ordering applied within each block and the ordering between blocks dictated by $<_{\mathbb{Q}}$ over $\{\, q_n \mid n < r_s\,\}$. Note that, in the construction, at any stage $s \geq 0$, if $I(n,s) \neq \emptyset$, then for any elements $k, m \in I(n,s)$, $k <^s_{\mathscr{L}} m \;\Leftrightarrow\; k <_{\mathbb{N}} m$. In other words the internal ordering of blocks always coincides with the natural ordering of $\mathbb{N}$. During the construction, for any stage s, and $k, m \in L_s$, if $k <^s_{\mathscr{L}} m$ then $k <^t_{\mathscr{L}} m$ for all $t \geq s$. Hence we will in general use $<_{\mathscr{L}}$ as shorthand for $<^s_{\mathscr{L}}$.

We choose some $d > 0$ as a default maximal block size for the construction.

Notation. During the construction we use the term *new* to refer to any finite set of numbers S which has not yet been enumerated into L at the present point in the construction and which is the minimal such set of cardinality $|S|$.

Block Rebuilding. At stage $s+1$ we may want to rebuild block $I(n,s)$ for some $n < r_s$. This means that there are distinct integers[6] $\hat{n} \geq 0$ and $\hat{m} > 0$ such that $|I(n,s)| = \hat{n}$ whereas we need $|I(n,s+1)| = \hat{m}$. We then proceed as follows according to whichever of the two cases below applies.

(a) $\hat{n} > \hat{m}$. In this case we search for the least labels $b_1, \ldots, b_{\hat{n}-\hat{m}} \geq r_s$ such that $q_n <_{\mathbb{Q}} q_{b_1} <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_{b_{\hat{n}-\hat{m}}} <_{\mathbb{Q}} q_a$ where $I(a,s)$ is the successor block to $I(n,s)$ in $\mathscr{L}_s$ (so that $q_n <_{\mathbb{Q}} q_a$) or q_a is simply any rational to the right of q_n if no such successor block exists. We define $S = \{\, b_j \mid 1 \leq j \leq \hat{n} - \hat{m} \,\}$ and $T = \{\, d \mid r_s \leq d \leq \max S \,\}$. Suppose that $k_1 <_{\mathscr{L}} \cdots <_{\mathscr{L}} k_{\hat{n}-\hat{m}}$ are the $\hat{n}-\hat{m}$ rightmost elements in $I(n,s)$. We remove $\{k_1, \ldots, k_{\hat{n}-\hat{m}}\}$ from $I(n,s)$ to obtain $|I(n,s+1)| = \hat{m}$ and proceed as follows. We firstly construct $I(b_1, s+1)$ by constructing it as the singleton block consisting of k_1 if $d = 1$ and otherwise we define it as $k_1 <_{\mathscr{L}} \hat{p} <_{\mathscr{L}} \cdots <_{\mathscr{L}} \hat{p} + d - 2$ (i.e. as a block of d elements) with $\{\hat{p}, \ldots, \hat{p} + d - 2\}$ a set of $d-1$ new numbers. We then proceed for each k_j such that $1 < j \leq \hat{n} - \hat{m}$ by constructing $I(b_j, s+1)$ in a similar fashion. Finally, for all $b \in T \setminus S$ we construct $I(b, s+1)$ using d new numbers. (Note that each $I(b, s+1)$ is inserted into $\mathscr{L}_{s+1}$ according to q_b's position under $<_{\mathbb{Q}}$ relative to $\{\, q_n \mid n < r_s \,\} \cup \{\, q_m \mid m \in T \,\} \setminus \{q_b\}$.)

(b) $\hat{n} < \hat{m}$. In this case, supposing that $I(n,s) = k_1 <_{\mathscr{L}} \cdots <_{\mathscr{L}} k_{\hat{n}}$ we choose a new set of $\hat{m} - \hat{n}$ numbers $\{\hat{p}, \ldots, \hat{p} + r\}$ where $r = \hat{m} - \hat{n} - 1$ and define $I(n, s+1) = k_1 <_{\mathscr{L}} \cdots <_{\mathscr{L}} k_{\hat{n}} <_{\mathscr{L}} \hat{p} <_{\mathscr{L}} \cdots <_{\mathscr{L}} \hat{p} + r$.

Note that, as this is the only rebuilding process applied during the construction we will be able to see, by inspection of the construction, that the following two conditions hold.

(i) For any $n \geq 0$, $\hat{m} > 0$ and stages $0 \leq s \leq t$, such that $|I(n,r)| \geq \hat{m}$ for all $s \leq r \leq t$, the block consisting of the $\hat{m}$ leftmost (i.e. least) elements in $I(n,t)$ is the same as that in $I(n,s)$.

(ii) For any $n, b, k \geq 0$ and stages $t > s \geq 0$, if k is removed from $I(n,s)$ at stage $s+1$ due to rebuilding and inserted into $I(b, s+1)$ as described above, then $k \in I(b,t)$. Note that this follows from (i) as k is the least number in $I(b, s+1)$. In other words any number can move from one block to another at most once.

[6] In fact $\hat{n} > 0$ for any such $n \leq s$ with the possible exception of $n = s$ at early stages of the construction.

The Diagonalisation Witness and Domain for R_e. For $s \geq e$, witness $m(e,s)$ for R_e is the construction's guess as to a number such that $\langle q_k, m(e,s)\rangle \notin U_j$, for all $0 \leq k, j \leq e$ such that U_j is the graph of a function $\mathbb{Q} \to \mathbb{N} \setminus \{0\}$. For all stages $s > e$, $m(e,s) = |I(e,s)| = F_s(e)$, where F_s is the s stage approximation to F. The set of *diagonalisation pairs* for index e is defined to be

$$P^e \;=\; \{\,(i,j) \;|\; 0 \leq i, j \leq e\,\}\,.$$

Thus $|P^e| = (1+e)^2$. Letting $x^e_0, \ldots, x^e_{(1+e)^2-1}$ be the computable ordering of P^e induced by the standard pairing function $\langle\cdot,\cdot\rangle$ we have the computable listing

$$D^e_0, \ldots, D^e_{2^{(1+e)^2}-1}$$

of all subsets of P^e defined using the canonical listing of finite sets $\{D_i\}_{i\in\mathbb{N}}$ specified above. (Note that $D^e_0 = \emptyset$ under this listing.) It is important to reiterate here that this means (by definition of the latter listing) that, for all $i, j \geq 0$,

$$D^e_i \subseteq D^e_j \quad\Rightarrow\quad i \leq j\,.$$

We now define the *diagonalisation domain* for R_e to be:

$$Z_e \;=\; X^e_0 \cup \ldots \cup X^e_{2^{(1+e)^2}-1}$$

where, for each $0 \leq i \leq 2^{(1+e)^2} - 1$, X^e_i is an interval of numbers associated with D^e_i such that (i) $|X^e_i| \geq |D^e_i| + 1$ (for reasons explained below) and (ii) for $i \neq 0$, $\min X^e_i > \max X^e_{i-1}$. To do this, for simplicity we define X^e_i such that $|X^e_i| = (1+e)^2 + 1$ for every $0 \leq i \leq 2^{(1+e)^2} - 1$ by defining:

$$X^e_i \;=\; \{i(1+e)^2 + (i+1), \ldots, (i+1)(1+e)^2 + (i+1)\}\,.$$

Accordingly Z_e is the interval $\{1, \ldots, |Z_e|\}$ partitioned by the X^e_i and having cardinality $|Z_e| = 2^{(1+e)^2}((1+e)^2 + 1)$.

The point here is that, at stage s we choose an index $i(e,s)$ for e in such a way that, $(k,j) \in D_{i(e,s)}$ if and only if there exists at most one number $n \in Z_e$ such that $\langle q_k, n\rangle \in U_{j,s}$. Since $|X^e_{i(e,s)}| \geq |D_{i(e,s)}| + 1$ we know that

$$X_{i(e,s)} \setminus \{\, r \;|\; r \in Z_e \;\&\; (\exists (k,j) \in D_{i(e,s)})[\,\langle q_k, r\rangle \in U_{j,s}\,]\,\} \;\neq\; \emptyset$$

so that we can define the s stage witness $m(e,s)$ to be a number in this set.

The Construction.

<u>*Stage $s = 0$.*</u>

Set $\mathscr{L}_0 = \langle L_0, <_{\mathscr{L}}\rangle$ with $L_0 = <_{\mathscr{L}} = \emptyset$. Define $I(n,0) = \emptyset$, $m(n,0) = 0$ and let $i(n,0)$ be undefined for all $n \geq 0$. Set $n_0 = r_0 = 0$.

<u>*Stage $s + 1$.*</u>

We suppose that n_s and r_s are such that $L_s = \mathbb{N}{\restriction}n_s$ and $\{\,n \;|\; I(n,s) \neq \emptyset\,\} =$

$\mathbb{N}\restriction r_s$. There are now $s+1$ substages $0 \leq e \leq s$ as follows.

Substage e.

Process requirement R_e as follows. Define $i(e, s+1) \in \mathbb{N}\restriction 2^{(1+e)^2}$ to be the index l satisfying

$$(k, j) \in D_l^e \;\Leftrightarrow\; |\{\, \langle q_k, r\rangle \mid r \in Z_e \,\} \cap U_{j,s+1}| \;\leq\; 1$$

for all $0 \leq k, j \leq e$. Let $i = i(e, s+1)$. If $i = 0$—i.e. if $D_i^e = \emptyset$—define $M^e(s+1) = X_i^e$. Otherwise define $M^e(s+1)$ as follows.

Notation. The *individual out-age* of $m \in X_i^e$ relative to any pair $(k, j) \in D_i^e$ at stage $s+1$—denoted $b^e(m, (k, j), s+1)$—is defined to be 0 if $\langle q_k, m\rangle \in U_{j,s+1}$ and otherwise is defined to be the greatest $0 < r \leq s+1$ such that $\langle q_k, m\rangle \notin U_{j,t}$ for all $(s+1) - r < t \leq s+1$. The *out-age* of $m \in X_i^e$ relative to D_i^e at stage $s+1$ is defined to be

$$a^e(m, s+1) \;=\; \min\{\, b^e(m, (k, j), s+1) \mid (k, j) \in D_i^e \,\}\,.$$

Define

$$M^e(s+1) = \{\, m \mid m \in X_i^e \;\&\; (\forall n \in X_i^e)[\, a^e(m, s+1) \geq a^e(n, s+1)\,]\,\}\,, \quad (1)$$

i.e. $M^e(s+1)$ contains all $m \in X_i^e$ of maximal out-age relative to D_i^e. (Note that, by construction, $a^e(m, s+1) = a^e(n, s+1) > 0$ for any $n, m \in M^e(s+1)$. Notice also that $M^e(s+1)$ contains precisely those numbers $m \in X_i^e$ for which it appears most likely that $\langle q_k, m\rangle \notin U_j$ for all $(k, j) \in D_i^e$.)

Define the $s+1$ stage witness $m(e, s+1) = \min M^e(s+1)$. If $m(e, s+1) = m(e, s)$ do nothing (so that $I(e, s+1) = I(e, s)$). Otherwise rebuild $I(e, s+1)$ from $I(e, s)$ as described under the *Block Building* above with[7] $\hat{n} = |I(e, s)|$ and $\hat{m} = m(e, s+1)$—so that $|I(e, s+1)| = m(e, s+1)$.

Ending Substage e.

If $e < s$ proceed to substage $e+1$. If $e = s$ define $\mathscr{L}_{s+1} = \langle L_{s+1}, <_{\mathscr{L}}\rangle$ as follows. Let $I(n, s+1) = I(n, s)$ for any labels $s < n < r_s$, i.e. for n such that $I(n, s)$ was a block in $\mathscr{L}_s$ but such that the block $I(n, s)$ was not rebuilt during one of the substages $0 \leq e \leq s$ during this stage. Set $r_{s+1} = \max\{\, n \mid I(n, s) \neq \emptyset \,\} + 1$, and $n_{s+1} = \max\{\, m \mid (\exists n < r_{s+1})[\, m \in I(n, s+1)\,]\,\} + 1$. Define $L_{s+1} = \mathbb{N}\restriction n_{s+1}$ and define $<_{\mathscr{L}}$ as dictated by the arrangement of the blocks $\{\, I(n, s+1) \mid n < r_{s+1} \,\}$ in $\mathscr{L}_{s+1}$ (and $<_{\mathscr{L}}$'s coincidence with the natural ordering inside each block). Proceed to stage $s+2$ in this case.

Verification.

Define $\mathscr{L} = \langle L, <_{\mathscr{L}}\rangle$ with $L = \bigcup_{s \geq 0} L_s$. Let

$$U \;=\; \{\, \langle e, m\rangle \mid \forall t(\exists s \geq t)[\, m(e, s) = m\,]\,\}$$

[7] For $e < s$, $|I(e, s)| = m(e, s)$. However for $e = s$ we have $m(e, s) = 0$ whereas it may be that $|I(e, s)| \neq 0$ due to previous rebuilding activity for the sake of some R_i with $i < e$.

and notice that U is a Π^0_2 set (as $m(e,s)$ is computable). Define

$$F(q_e) \;=\; \mu m[\,\langle e,m\rangle \in U\,]\,. \tag{2}$$

Note that $F(q_e)$ is defined for all e as $m(e,s)$ is defined as an element of the finite set Z_e for all $s > e$. Notice also that $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ (and that the construction of F is Δ^0_3 as witnessed by (2)).

We see by inspection that $L = \mathbb{N}$ and that $\mathscr{L}$ has order type $\sum\{\,F(q) \mid q \in \mathbb{Q}\,\}$. Indeed by construction $n \in L_{n+1} \subseteq L$. Moreover n can be moved from one block $I(b,s)$ into another block $I(a,s+1)$ at stage $s+1$ only via the *Block Rebuilding* process. However in this case $n \in I(a,t)$ for all $t \geq s+1$ as explained in remark (ii) on page 5. Thus n changes blocks at most once. Now consider $e \geq 0$. Let s_e be a stage such that $m(e,s_e) = F(q_e)$ and such that $m(e,s) \geq m(e,s_e)$ for all $s \geq s_e$. Then $|I(e,s_e)| = F(q_e)$ and $\{\,s \mid |I(e,s)| = F(q_e)\,\}$ is infinite. Moreover, as stated in observation (i) on page 5, the leftmost block of $F(q_e)$ is preserved in $I(e,t)$ for all $t \geq s_e$. I.e. $I(e)$ is well defined as a maximal block with cardinality $F(q_e)$.

For $e \geq 0$ as above, define $i(e)$ to be the index satisfying

$$(k,j) \in D^e_{i(e)} \;\Leftrightarrow\; |\{\,\langle q_k, r\rangle \mid r \in Z_e\,\} \cap U_j| \leq 1$$

for all $0 \leq k,j \leq e$. Let $t_e > s_e$ be a stage such that

$$|\{\,\langle q_k, r\rangle \mid r \in Z_e\,\} \cap U_{j,s}| \;\leq\; 1$$

for all $(k,j) \in D^e_i$ and $s \geq t_e$. Then, by definition, at any such stage s, $D^e_{i(e)} \subseteq D^e_{i(e,s)}$ and so $i(e) \leq i(e,s)$. For each $0 \leq j \leq e$ define[8]

$$E_j \;=\; \{\,\langle q_k, r\rangle \mid r \in Z_e \;\&\; k \leq e \;\&\; (k,j) \notin D^e_{i(e)}\,\} \cap U_j \tag{3}$$

By Lemma 1, there are infinitely many stages s such that $E_j \subseteq U_{j,s}$ for all $0 \leq j \leq e$. Moreover, at each such stage $s \geq t_e$, $i(e) = i(e,s)$ by definition of the construction. On the other hand, as $|X_{i(e)}| > |D_{i(e)}|$, we see that $X_{i(e)} = S \cup T$ with $S \neq \emptyset$ and $S \cap T = \emptyset$ where

$$T \;=\; \{\,r \mid (\exists (k,j) \in D^e_{i(e)})[\,\langle q_k, r\rangle \in U_j\,]\,\} \cap X_{i(e)}$$

and $S = X_{i(e)} \setminus T$. By definition of S there is a stage $\hat{t}_e \geq t_e$ such that, for all $s \geq \hat{t}_e$, and for every $r \in S$, there is no $(k,j) \in D_{i(e)}$ such that $\langle q_k, r\rangle \in U_{j,s}$. For each $r \in S$, let the *in-age* of r relative to $D_{i(e)}$ be the greatest stage s such that $\langle q_k, r\rangle \in U_{j,s}$ for some $(k,j) \in D_{i(e)}$ if such a stage exists, and otherwise define the *in-age* of r to be 0. Define $M \subseteq S$ to be the elements of S of least in-age relative to $D_{i(e)}$, and choose m to be the least number in M. (Note that by definition $M = X_{i(e)}$ if $i(e) = 0$.) Now, for each $0 \leq j \leq e$ define

$$\widehat{E}_j \;=\; \{\,\langle q_k, r\rangle \mid r \in X_{i(e)} \;\&\; k \leq e \;\&\; (k,j) \in D^e_{i(e)}\,\} \cap U_j\,.$$

[8] We could also simply define $E_j = \{\,\langle q_k, r\rangle \mid r \in Z_e \;\&\; k \leq e\,\} \cap U_j$ with the same result.

By Lemma 1 we know that there exists a stage $u_e \geq \hat{t}_e$ such that, not only $E_j \subseteq U_{j,u_e}$ for all $0 \leq j \leq e$ (with E_j defined as in (3)), so that $i(e,u_e) = i(e)$, but also $\widehat{E}_j \subseteq U_j$ for each such j so that, by definition $m(e,u_e) = m$ and moreover, $m(e,s) = m$ for all stages $s \geq u_e$ such that $i(e,s) = i(e)$. Accordingly, letting $m(e) = m$ we see that $m(e)$ is the witness for R_e at every such stage s and $|I(e,s)| = m(e)$. Moreover, at every stage $s \geq s_e$ such that $i(e,s) \neq i(e)$, we know that $i(e,s) > i(e)$ as $u_e \geq s_e$. However this implies that $m(e,s) > m(e)$ at every such stage s as $m(e,s) \in X_{i(e,s)}$ and $\min X_{i(e,s)} > \max X_{i(e)} \geq m(e)$. It follows that $F(q_e) = |I(e)| = m(e)$.

Note 4. Suppose that $\mathscr{B}$ is a linear ordering and $\iota : \mathscr{B} \cong \mathscr{L}$ is an isomorphism. Suppose also that $\widehat{F} : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ is a maximal block function of $\mathscr{B}$ and that $\widehat{I}(0), \widehat{I}(1), \widehat{I}(2), \ldots$ is an assignment of $\widehat{F}$ to $\mathscr{B}$. Now note that we have a listing of labels of $\mathscr{L}$,

$$m_0, m_1, m_2, \ldots$$

such that

$$\iota : \widehat{I}(j) \cong I(m_j)$$

(i.e. $I(m_j)$ is the isomorphic image of $\widehat{I}(j)$ under ι) for all $j \geq 0$. Moreover there must be infinitely many labels j of $\mathscr{B}$ such that $m_j \geq j$. Indeed, suppose otherwise so that for some l, for all $j \geq l$ we have $m_j < j$. Choose $m = \max\{\, m_j \mid j < l \,\} \cup \{l\} + 1$. Then, under our assumption,

$$\iota^*(\{\, n \mid n \leq m \,\}) \subseteq \{\, n \mid n < m \,\}$$

where ι^* is the map over labels induced by ι. Thus ι^* is not one-one. This contradicts the fact that ι is an isomorphism. We therefore conclude that there are infinitely many pairs of labels (k,e) with $k \leq e$ such that $\iota : \widehat{I}(k) \cong I(e)$.

Choose any $\mathscr{B}$, ι, $\widehat{F}$ and assignment $\widehat{I}(0), \widehat{I}(1), \widehat{I}(2), \ldots$ as in Note 4. Consider any index $j \geq 0$ and suppose that U_j is the graph of a function G_j with domain $\mathbb{Q}$. As above, choose $k \geq j$ such that $\iota : \widehat{I}(k) \cong I(e)$ for some $e \geq k$. Now, by definition of the construction, $(k,j) \in D_{i(e)}$. However this implies that

$$G_j(q_k) \neq m(e) = F(q_e) = |I(e)| = |\widehat{I}(k)|\,.$$

Note in the above argument that the choice of $\widehat{F}$, and of its assignment to $\mathscr{B}$, as also of the isomorphism $\iota : \mathscr{B} \cong \mathscr{L}$, was in each case arbitrary. Notice also that the same observation holds for the choice of the index $j \geq 0$, and of the linear ordering $\mathscr{B} \cong \mathscr{L}$. We can thus conclude that, for *any* Π^0_2 function $G : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ and *any* $\mathscr{B} \cong \mathscr{L}$, $\mathscr{B}$ does not have order type $\tau = \sum\{\, G(q) \mid q \in \mathbb{Q} \,\}$. □

Note 5. We can choose any $p \geq 0$ and replace $F : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ by $F : \mathbb{Q} \to \mathbb{N} \setminus \{0, \ldots, p\}$ in the statement of Theorem 2, by a simple adjustment of the proof, so ensuring that $\mathscr{L}$ contains no maximal blocks of size p or less. We can also clearly force F to be injective (so making $\mathcal{L}$ rigid). For example if we define each Z_e as before but such that $\min Z_{e+1} > \max Z_e$ we obtain F strictly increasing.

We conclude by noting another application of our proof technique and how this yields an alternative proof[9] of Theorem 2 via the work of either Kach or (Kenneth) Harris. Indeed, a straightforward adaptation of the framework of the proof of Theorem 2 can be applied to show that there exists a **0**-*limitwise monotonic* set[10] $S \subseteq \mathbb{N} \setminus \{0,1\}$ such that S is the range of *no* Π^0_1 function[11] G (with domain $\mathbb{N}$). Relativising this result we obtain a $\mathbf{0'}$-*limitwise monotonic* set S such that the *shuffle sum* of S derived via the proof of Proposition 2.1 of [Kac08] and the η-*representation* of S derived via the proof of Theorem 3.3 of [Har08] are both examples of η-like computable linear orderings having no isomorphic copy with Π^0_2 maximal block function[12].

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9 The author is grateful to the anonymous referees for pointing this out.

10 A function H is $\boldsymbol{a}$-*limitwise monotonic* if there is an $\boldsymbol{a}$-computable function h such that, for all x, (i) $H(x) = \lim_{s\to\infty} h(x,s)$, and (ii) for all s, $h(x,s) \leq h(x,s+1)$. A set S is $\boldsymbol{a}$-*limitwise monotonic* if it is the range of an $\boldsymbol{a}$-*limitwise monotonic* function.

11 A $\emptyset'$ priority argument shows that, if G has infinite range, then there exists injective Π^0_1 $\widehat{G}$ with the same range. We use this to apply an adapted version of Note 4.

12 The existence of such a function F' (viewed as having domain $\mathbb{N}$) would yield an obvious contradiction—in the shuffle sum case witnessed by F' itself and in the η-*representation* case witnessed by the function with graph $G(F') \setminus \{\, \langle n, 1\rangle \mid n \in \mathbb{N}\,\}$.