The invariant degrees, randomness, and semi-measures

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Journées Calculabilité 2014
Montpellier
28 April 2014

Motivation

In computability theory, one major concern is to determine which problems are solvable by Turing machines and which ones are not.

- For a given set of natural numbers S, is there an effective procedure for determining membership in S?

If a given problem is shown to be effectively unsolvable, we can further ask: Just how unsolvable is it?

In fact, there are a number of hierarchies for classifying the difficulty of solving various problems.

An alternative approach

In these investigations of the solvability/unsolvability of problems, the computations are carried out by Turing machines (often equipped with an oracle).

What picture would emerge if instead we were to work with some model of probabilistic Turing machine?

Is there a reasonable degree structure for studying something like the degrees of solvability or the degrees of unsolvability of a class of problems with respect to probabilistic computation?

As we will see shortly, the degree structure I will discuss today can be seen as measuring the degree of probabilistic *solvability* of a given problem.

Which problems?

Which problems should we attempt to solve probabilistically?

If the problems we consider are those of determining the members of some subset of the natural numbers, the degrees of solvability are uninteresting.

Theorem (Sacks)

A sequence is computable with positive probability if and only if it is computable.

In the context of probabilistic computation, to get a more interesting degree structure, we need to investigate the degree of solvability of collections of sequences.

Turing invariant subsets of 2^ω

The collections of sequences we will consider are Turing invariant collections:

$\mathcal{A} \subseteq 2^\omega$ is *Turing invariant* for every $X \in \mathcal{A}$ and $Y \in 2^\omega$, if X and Y have the same Turing degree, then $Y \in \mathcal{A}$.

The goal of today's talk is to discuss recent work on a degree structure that measures the degree of solvability of Turing invariants subsets of 2^ω in terms of probabilistic computation.

We call this degree structure the *invariant degrees*.

Outline of the talk

1. Probabilistic Turing computation
2. Negligibility and semi-measures
3. The invariant degrees
4. V'yugin's technique for constructing semi-measures

1. Probabilistic Turing computation

Two approaches to probabilistic computation

One standard definition of a probabilistic Turing machine is a non-deterministic Turing machine whose transitions are chosen according to some probability distribution.

Alternatively, one can define a probabilistic machine to be an oracle Turing machine with some algorithmically random sequence as an oracle.

Key idea: For the purposes of computing a sequence or some sequence in a fixed class collection *with positive probability*, these two approaches are equivalent.

Ingredient #1: Turing functionals

The first ingredient for the model of probabilistic computation that we'll be working with is the notion of a Turing functional.

Definition

A *Turing functional* $\Phi : 2^\omega \to 2^\omega$ is a computably enumerable set of pairs of strings (σ, τ) such that if $(\sigma, \tau), (\sigma', \tau') \in \Phi$ and $\sigma \preceq \sigma'$, then $\tau \preceq \tau'$ or $\tau' \preceq \tau$.

If $\Phi(B){\downarrow} = A$, then we say that A is *Turing reducible* to B, denoted $A \leq_T B$.

Moreover, A is *Turing equivalent* to B (or A has the same *Turing degree* as B), denoted $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$.

Ingredient #2: Algorithmically random sequences

The second ingredient for our model of probabilistic computation is the notion of an algorithmically random sequence.

There are a number of different definitions of algorithmic random sequences, many of which are not equivalent (but they differ from one another on a set of measure zero).

One common form of many of these definitions is that a sequence is held to be random if it avoids a certain kind of null set.

The Lebesgue measure on 2^ω

Given $\sigma \in 2^{<\omega}$,

$$[\![\sigma]\!] := \{X \in 2^\omega : \sigma \prec X\}.$$

These are the basic open sets of 2^ω.

The Lebesgue measure on 2^ω is defined by

$$\lambda([\![\sigma]\!]) = 2^{-|\sigma|}.$$

λ can be extended to all Borel subsets of 2^ω in the standard way.

Martin-Löf randomness

Definition

- A *Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly effectively open subsets of 2^ω such that for each i,

$$\lambda(\mathcal{U}_i) \leq 2^{-i}.$$

- X *passes* the Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.
- $X \in 2^\omega$ is *Martin-Löf random*, denoted $X \in \mathrm{MLR}$, if X passes every Martin-Löf test.

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$$\text{2MLR} \subsetneq \text{W2R} \subsetneq \text{MLR} \subsetneq \text{SR} \subsetneq \text{KR}$$

Putting the ingredients together

Let $\mathcal{S} \subseteq 2^\omega$.

Problem P: Probabilistically compute a member of $\mathcal{S}$.

A solution to the problem P: There is a Turing functional Φ such that

$$\lambda(\{X \in 2^\omega : \Phi(X) \in \mathcal{S}\}) > 0.$$

Equivalently, we can require

$$\lambda(\{X \in \mathrm{MLR} : \Phi(X) \in \mathcal{S}\}) > 0.$$

In fact, we can replace MLR with any of 2MLR, W2R, SR, KR, or *any* reasonable notion of algorithmic randomness.

Taking stock

We've now laid out the model of probabilistic computability in terms of which the invariant degrees are defined.

We now need a way to show that one class of sequences is more solvable than another.

To define such an order, the key notion we will draw upon is that of *negligibility*.

2. Negligibility and semi-measures

The definition of negligibility

A subset $\mathcal{A}$ of 2^ω is *negligible* if we cannot compute some member of $\mathcal{A}$ with positive probability.

That is,

$$\lambda\big(\{X \in 2^\omega : (\exists Y \in \mathcal{A})[Y \leq_T X]\}\big) = 0.$$

We can also provide a useful equivalent formulation of negligibility in terms of left-c.e. semi-measures.

Left-c.e. semi-measures

A *semi-measure* $\rho : 2^{<\omega} \to [0,1]$ satisfies

- $\rho(\varnothing) = 1$ and
- $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$ for every $\sigma \in 2^{<\omega}$.

A semi-measure ρ is *left-c.e.* if each value $\rho(\sigma)$ is the limit of a non-decreasing computable sequence of rationals, uniformly in σ.

Semi-measures and Turing functionals

For $\sigma \in 2^{<\omega}$, we define $\Phi^{-1}(\sigma) := \{X \in 2^\omega : \exists n\, (X{\upharpoonright}n, \sigma) \in \Phi\}$.

Proposition

(i) *If Φ is a Turing functional, then λ_Φ, defined by*

$$\lambda_\Phi(\sigma) = \lambda(\Phi^{-1}(\sigma))$$

for every $\sigma \in 2^{<\omega}$, is a left-c.e. semi-measure.

(ii) *For every left c.e. semi-measure ρ, there is a Turing functional Φ such that $\rho = \lambda_\Phi$.*

A universal semi-measures

Levin proved the existence of a *universal* left-c.e. semi-measure.

A left-c.e. semi-measure M is universal if for every left-c.e. semi-measure ρ, there is some $c \in \omega$ such that

$$\rho(\sigma) \leq c \cdot M(\sigma)$$

for every $\sigma \in 2^{<\omega}$.

Defining negligibility in terms of semi-measures

Let M be a universal left-c.e. semi-measure.

Let $\overline{M}$ be the largest measure such that $\overline{M} \leq M$, which can be seen as a universal measure.

Proposition

$\mathcal{S} \subseteq 2^\omega$ is negligible if and only if $\overline{M}(\mathcal{S}) = 0$.

Proof idea: Use the correspondence between Turing functionals and left-c.e. semi-measures and the fact that M multiplicatively dominates all left-c.e. semi-measures to show

$$\overline{M}(\mathcal{S}) = 0 \text{ if and only if } \lambda\Big(\bigcup_{i\in\omega} \Phi_i^{-1}(\mathcal{S})\Big) = 0$$

for every $\mathcal{S} \subseteq 2^\omega$ (where $(\Phi_i)_{i\in\omega}$ is an effective enumeration of all Turing functionals).

3. The invariant degrees

The ordering $\leq_I$

Given Turing invariant classes $\mathcal{A}, \mathcal{B} \subseteq 2^\omega$, we define

$$\mathcal{A} \leq_I \mathcal{B} \;\Leftrightarrow\; \mathcal{A} \setminus \mathcal{B} \text{ is negligible.}$$

What does this mean?

If $\mathcal{A} \setminus \mathcal{B}$ is negligible, then

$$\overline{M}(\mathcal{A}) = \overline{M}(\mathcal{A} \cap \mathcal{B}) + \overline{M}(\mathcal{A} \setminus \mathcal{B}) = \overline{M}(\mathcal{A} \cap \mathcal{B}).$$

In general, $\mathcal{A} \leq_I \mathcal{B}$ tells us that any probabilistic algorithm that produces a member of $\mathcal{A}$ with probability $> p$ for some $p \in (0,1)$ also produces a member of a member of $\mathcal{B}$ with probability $> p$.

The collection of invariant degrees

As usual, we write $\mathcal{A} \equiv_I \mathcal{B}$ whenever we have $\mathcal{A} \leq_I \mathcal{B}$ and $\mathcal{B} \leq_I \mathcal{A}$.

For Turing invariant $\mathcal{A} \subseteq 2^\omega$, the invariant degree of $\mathcal{A}$ is

$$\deg_I(\mathcal{A}) = \{\mathcal{B} \subseteq 2^\omega : \mathcal{B} \text{ is Turing invariant and } \mathcal{A} \equiv_I \mathcal{B}\}.$$

For an arbitrary $\mathcal{S} \subseteq 2^\omega$, we will write $\deg_I(\mathcal{S})$ as shorthand for $\deg_I((\mathcal{S})^{\equiv_T})$.

The collection of invariant degrees is denoted $\mathscr{D}_I$.

A few observations

Observation 1: The collection of Turing invariant subsets of 2^ω forms a Boolean algebra under the operations of $\cup$, $\cap$, and ^{c}.

Observation 2: $\mathcal{A} \equiv_I \mathcal{B}$ if and only if $\overline{M}(\mathcal{A} \, \Delta \, \mathcal{B}) = 0$.

Observation 3: The previous two observations imply that $(\mathscr{D}_I, \overline{M})$ forms a measure algebra.

General properties of $\mathscr{D}_I$

- The bottom degree of $\mathscr{D}_I$ consists of all Turing invariant negligible sets.
- The top degree of $\mathscr{D}_I$ consists of all sets of the form $2^\omega \setminus \mathcal{A}$, where $\mathcal{A}$ is Turing invariant and negligible.
- $\mathscr{D}_I$ is atomic: There exists a Turing invariant $\mathcal{A} \subseteq 2^\omega$ such that for any Turing invariant $\mathcal{B} \subseteq \mathcal{A}$, either $\mathcal{B}$ is negligible or $\mathcal{A} \equiv_I \mathcal{B}$.
  - The collection of computable sets forms an atom of $\mathscr{D}_I$ (Levin, V'yugin).
  - The collection of sequences Turing equivalent to a Martin-Löf random sequence forms an atom of $\mathscr{D}_I$ (Levin, V'yugin).

Levin's question

Let $\mathcal{C}$ denote the collection of computable sequences. Do we have

$$\deg_I(\mathrm{MLR} \cup \mathcal{C}) = \deg_I(2^\omega)?$$

In other words, is $2^\omega \setminus (\mathrm{MLR} \cup \mathcal{C})$ negligible?

A negative answer was first provided by V'yugin, who proved a very general result.

Theorem (V'yugin)

For every $\epsilon > 0$, there is a probabilistic algorithm that produces with probability greater than $1 - \epsilon$ a non-computable sequence that does not compute any Martin-Löf random sequence.

In the final part of the talk, I will discuss the technique V'yugin used to prove this result.

An alternative answer

We can also answer Levin's question by means of the following result:

Theorem (Kautz, Kurtz)

Every 2-random computes a 1-generic.

Let $\mathcal{G}$ denote the collection of 1-generic sequences.

- By the above result, $\mathcal{G}$ (and hence $(\mathcal{G})^{\equiv_T}$) is non-negligible.
- Moreover, $(\mathcal{G})^{\equiv_T} \cap (\mathrm{MLR})^{\equiv_T} = \emptyset$
- Thus, $\deg_I(\mathrm{MLR})$ and $\deg_I(\mathcal{G})$ are incomparable with respect to $\leq_I$.

What about other notions of randomness?

Proposition (Bienvenu, Hölzl, Porter)

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Proof.

- Since 2MLR $\subseteq$ W2R $\subseteq$ MLR $\subseteq$ SR, it follows that 2MLR $\leq_I$ W2R $\leq_I$ MLR $\leq_I$ SR.
- If $X \in$ MLR $\setminus$ 2MLR, X cannot be computed by a 2-random (by the XYZ-theorem, if X is computable from a 2-random, it must be 2-random). Thus MLR $\setminus$ 2MLR is negligible.

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- $(\mathsf{W2R} \setminus \mathsf{2MLR}) \subseteq (\mathsf{MLR} \setminus \mathsf{2MLR})$, which implies that $\mathsf{W2R} \setminus \mathsf{2MLR}$ is negligible.

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- If $X \in \mathsf{MLR} \setminus \mathsf{2MLR}$, X cannot be computed by a 2-random (by the XYZ-theorem, if X is computable from a 2-random, it must be 2-random). Thus $\mathsf{MLR} \setminus \mathsf{2MLR}$ is negligible.
- $(\mathsf{W2R} \setminus \mathsf{2MLR}) \subseteq (\mathsf{MLR} \setminus \mathsf{2MLR})$, which implies that $\mathsf{W2R} \setminus \mathsf{2MLR}$ is negligible.
- Every $X \in \mathsf{SR} \setminus \mathsf{MLR}$ has high degree, but no 3-random can compute a sequence of high degree. Thus $\mathsf{SR} \setminus \mathsf{MLR}$ is negligible.

One outlier: Kurtz randomness

Proposition (Bienvenu, Hölzl, Porter)

(1) MLR $<_I$ KR *and* $\mathcal{G} <_I$ KR

(2) KR $\equiv_I$ HI $\equiv_I$ WG, *where* HI *consists of all sequences of hyperimmune Turing degree and* WG *consists of all weakly 1-generic sequences.*

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- The Turing degrees of weakly 1-generics are precisely the hyperimmune degrees.

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To establish (2) we use the following facts:

- The Turing degrees of weakly 1-generics are precisely the hyperimmune degrees.
- Every weakly 1-generic is Kurtz random.
- If $X \in \mathrm{KR} \setminus \mathrm{HI}$, then X has hyperimmune-free degree and is thus weakly 2-random. This implies that $X \in \mathrm{W2R} \setminus \mathrm{2MLR}$, which we have already shown is negligible.

Diagonal non-computability

Recall:

- A total function f is a DNC function if $f(e) \neq \phi_e(e)$ for every e, where $(\phi_e)_{e\in\omega}$ is the collection of partial computable functions.
- $X \in 2^\omega$ has DNC Turing degree if X computes a DNC function.

Every $X \in \mathrm{MLR}$ has DNC Turing degree, and hence it follows that $\mathrm{MLR} \leq_I \mathrm{DNC}$.

What about the converse?

Theorem (Bienvenu, Patey)

$\mathrm{DNC} \setminus \mathrm{MLR}$ *is non-negligible. In particular,* $\mathrm{MLR} <_I \mathrm{DNC}$.

4. V'yugin's technique for constructing semi-measures

Revisiting V'yugin's theorem

Theorem (V'yugin)

For every $\epsilon > 0$*, there is a probabilistic algorithm that produces with probability greater than* $1 - \epsilon$ *a non-computable sequence that does not compute any Martin-Löf random sequence.*

An analysis of V'yugin's proof shows that he proves the following.

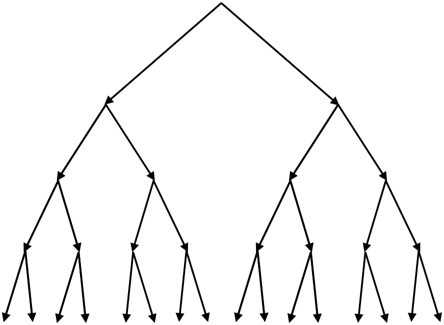
Theorem

For every $\epsilon > 0$*, there is a* Π^0_1 *class* $\mathcal{P}$ *and a left-c.e. semi-measure* ρ *such that*

- $\overline{\rho}(\mathcal{P}) > 1 - \epsilon$*, and*
- *no member of* $\mathcal{P}$ *computes any Martin-Löf random sequence.*

The general idea 1

- Identify $2^{<\omega}$ with a directed graph, where the arrows point in only one direction, away from the root.



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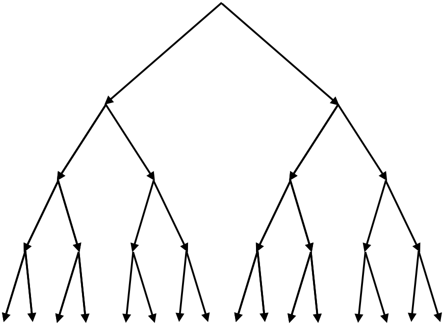
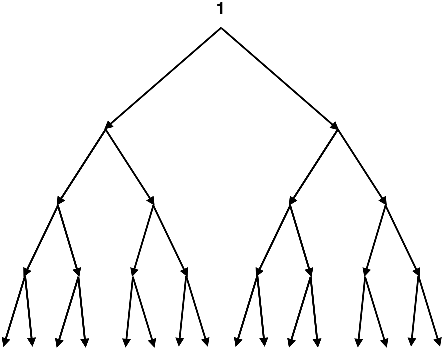
- Identify $2^{<\omega}$ with a directed graph, where the arrows point in only one direction, away from the root.
- Identify a semi-measure on 2^{ω} with a network flow on the directed graph.
  - Give the empty string ε amount of flow equal to 1.
  - The sum of the flow into the two extensions $\sigma 0$ and $\sigma 1$ of a node σ cannot exceed the amount of flow into σ.
  - Flow can leak out of the system.

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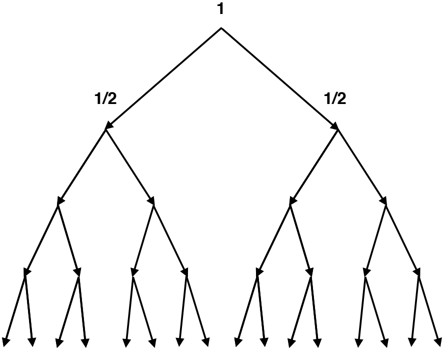
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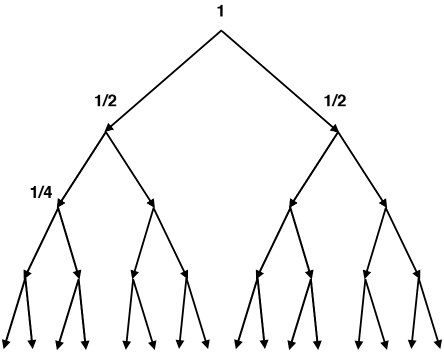
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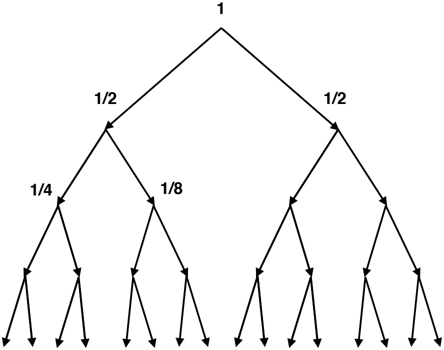
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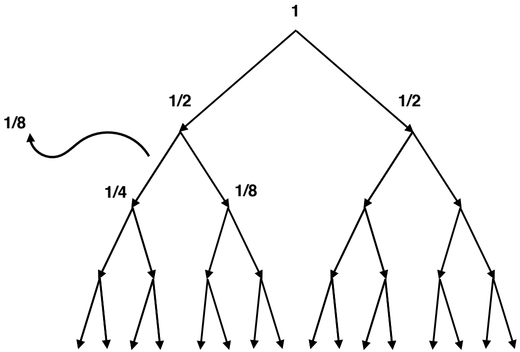
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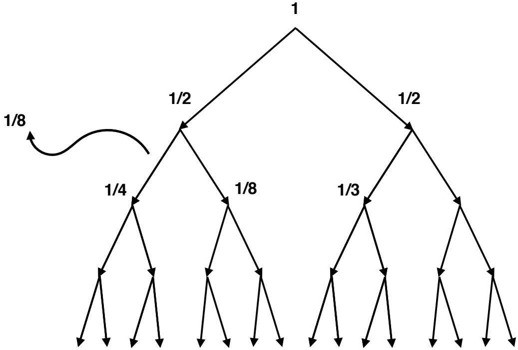
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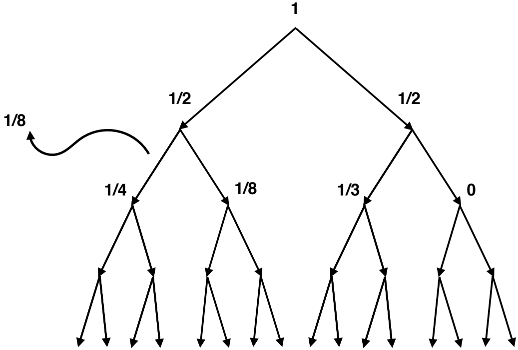
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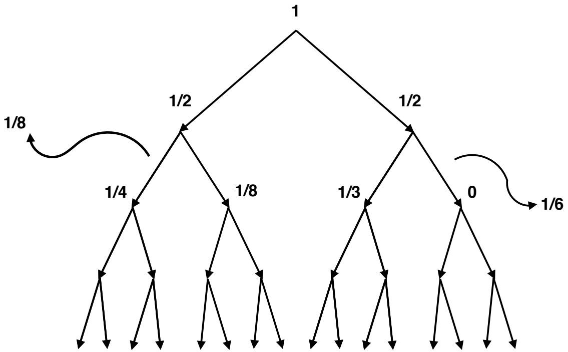
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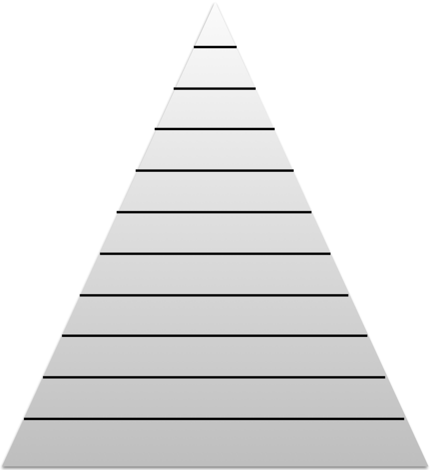
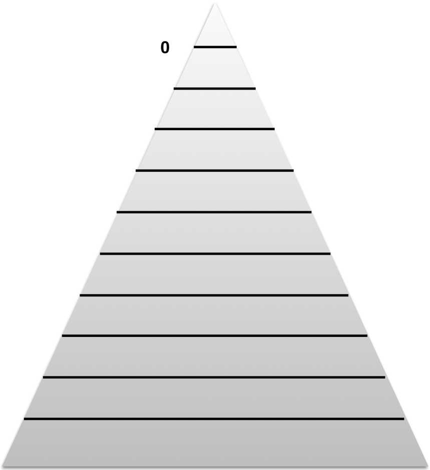
  - Give the empty string ε amount of flow equal to 1.
  - The sum of the flow into the two extensions $\sigma 0$ and $\sigma 1$ of a node σ cannot exceed the amount of flow into σ.
  - Flow can leak out of the system.
- We have an infinite number of requirements coded by pairs (i, n) which are satisfied by a string σ if $|\Phi_i^{\sigma}| > f(\sigma, n)$ for some fixed computable function f.

The general idea 2

- Associate to each level of $2^{<\omega}$ a requirement in such a way that every requirement occurs infinitely often.

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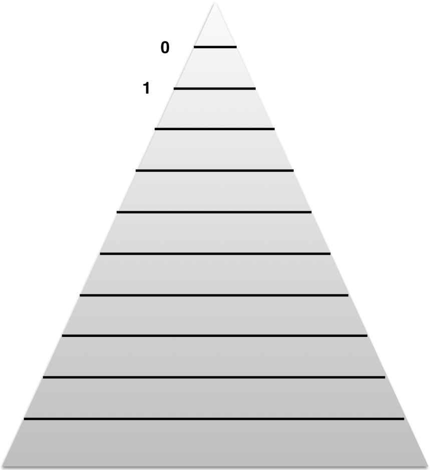

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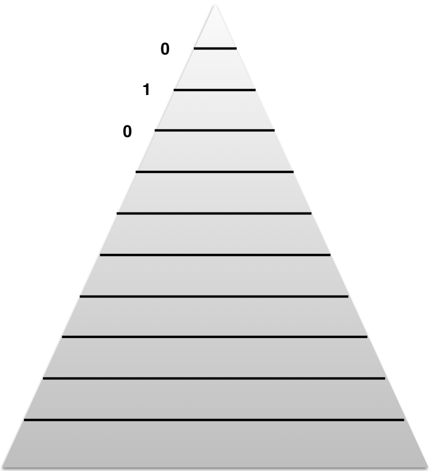
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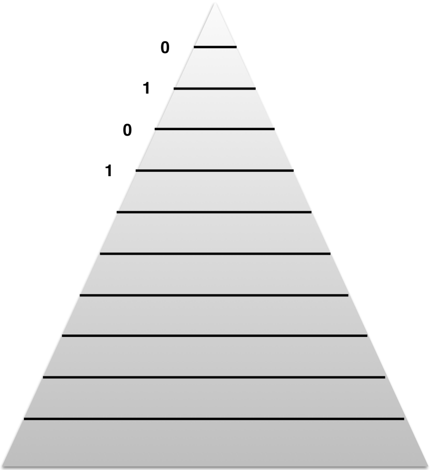
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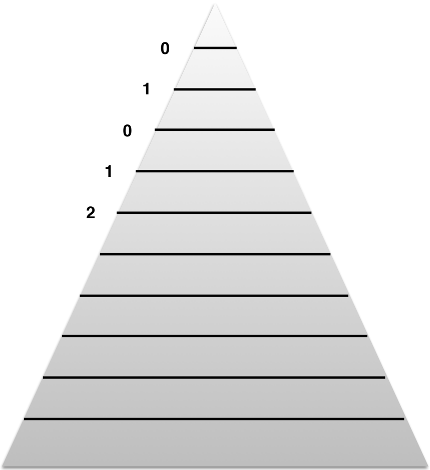
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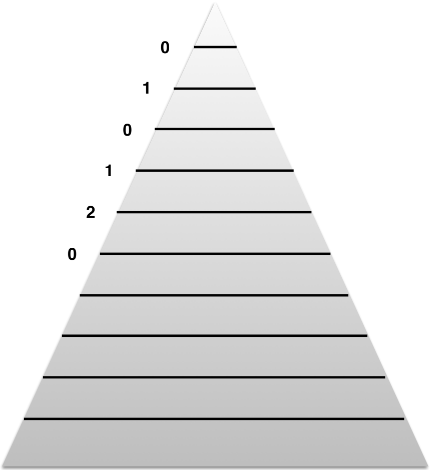
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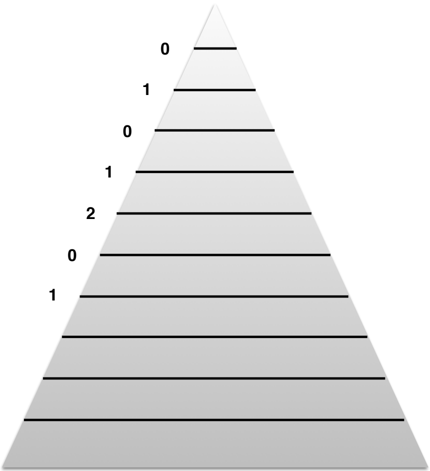

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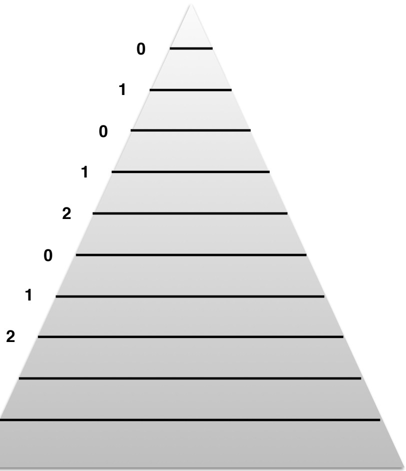
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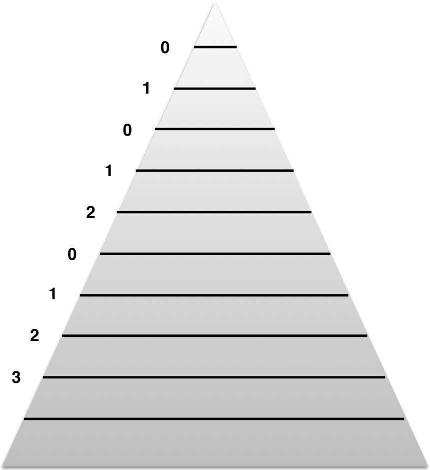
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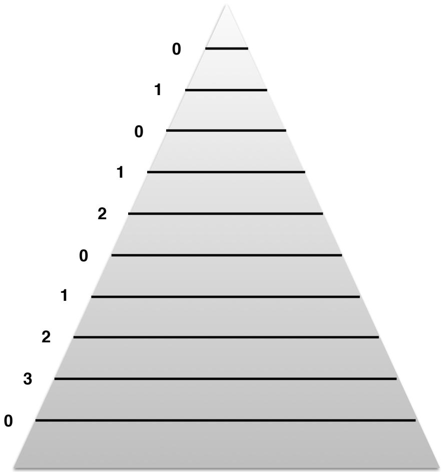
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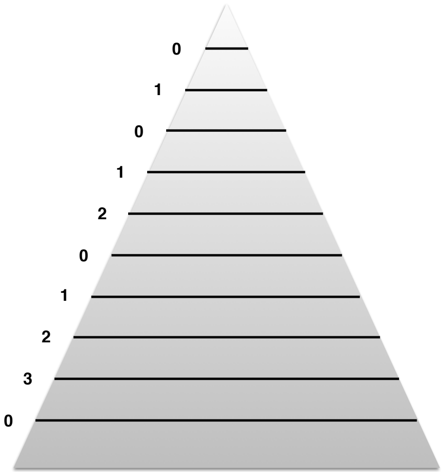
The general idea 2

- Associate to each level of $2^{<\omega}$ a requirement in such a way that every requirement occurs infinitely often.
- The general strategy is to block flow at nodes where a requirement first appears; if we can satisfy that requirement, we mount an additional edge on $2^{<\omega}$ and pass the blocked flow through that edge.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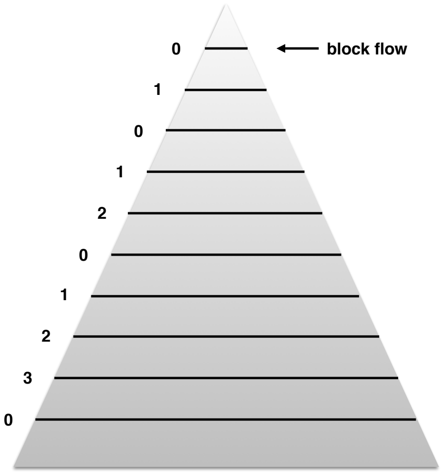
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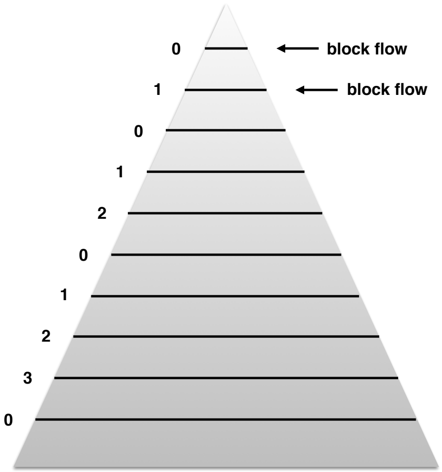
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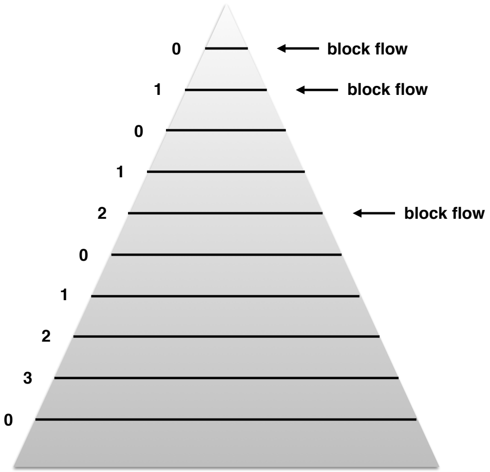
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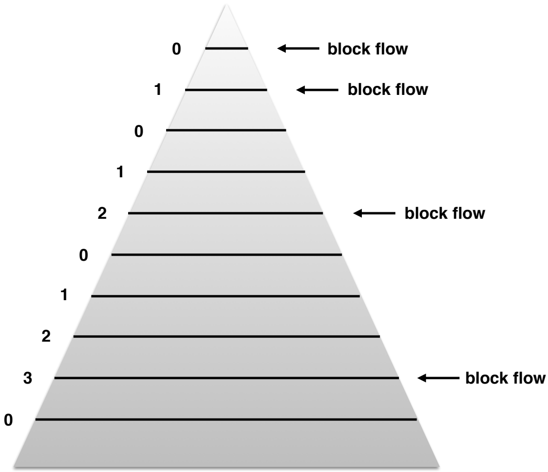
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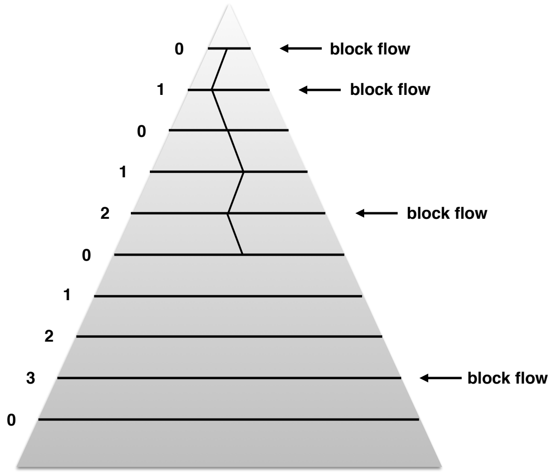
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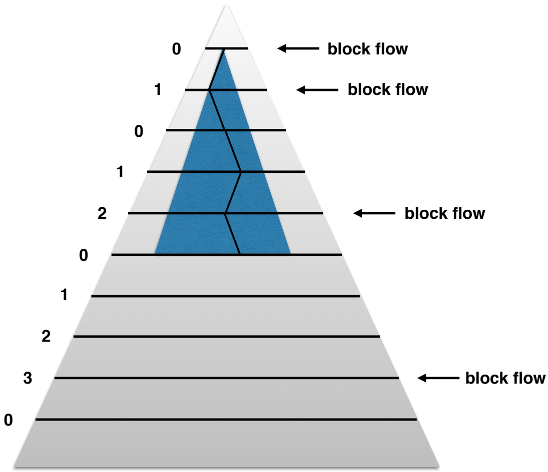

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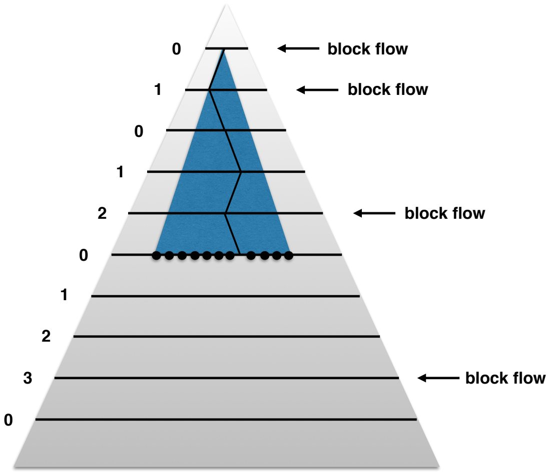

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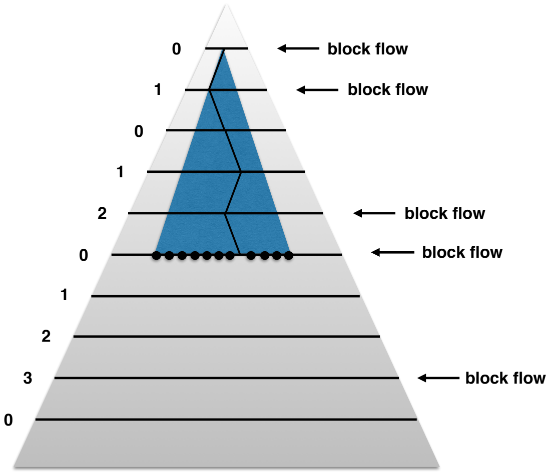
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The general idea 2

- Associate to each level of $2^{<\omega}$ a requirement in such a way that every requirement occurs infinitely often.
- The general strategy is to block flow at nodes where a requirement first appears; if we can satisfy that requirement, we mount an additional edge on $2^{<\omega}$ and pass the blocked flow through that edge.
- Some nodes will have all flow blocked from them; the key to the construction is to ensure that this doesn't happen too often.
- At the same time, we have to ensure that the remaining paths, whose initial segments never have all of their flow completely blocked, cannot compute a Martin-Löf random sequence.

Applications to the invariant degrees

Using more complicated versions of this technique, V'yugin proves the following results:

Theorem (V'yugin)

$\mathscr{D}_I$ has countably many atoms.

Theorem (V'yugin)

The union of the atoms of $\mathscr{D}_I$ is not I-equivalent to the top element.

Improving V'yugin's theorem

In joint work with Rupert Hölzl, we obtained the following:

Theorem (Hölzl, Porter)

For every $\epsilon > 0$, there is a Π^0_1 class $\mathcal{P}$ and a left-c.e. semi-measure ρ such that

- $\overline{\rho}(\mathcal{P}) > 1 - \epsilon$, *and*
- *no member of $\mathcal{P}$ computes any sequence of DNC degree.*

Idea: Modify the original V'yugin construction while making use of a characterization of Simpson and Stephan of sequences of DNC degree in terms of f-randomness.

Thank you for your attention.