Computational Complexity of the GPAC

Amaury Pouly
Joint work with Olivier Bournez and Daniel Graça

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Outline

The case of discrete computations

Many models:

- Recursive functions
- Turing machines
- λ-calculus
- circuits
- ...

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Church Thesis

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- Church Thesis for analog computers ? $\Rightarrow$ No (GPAC $\neq$ BSS)
- Comparison with digital models of computation ? $\Rightarrow$ How ?
- What is a “reasonable” model ? $\Rightarrow$ Unclear

GPAC

General Purpose Analog Computer

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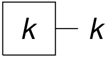
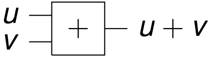
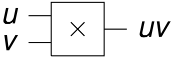


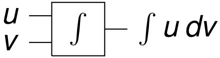
A constant unit



An adder unit



An multiplier unit



An integrator unit

GPAC: beyond the circuit approach

Theorem

y is generated by a GPAC iff it is a component of the solution $y = (y_1, \dots, y_d)$ of the Polynomial Initial Value Problem (PIVP):

$$\begin{cases} y' &= p(y) \\ y(t_0) &= y_0 \end{cases}$$

where p is a vector of polynomials.

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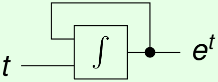
Remark

Other point of view: continuous dynamical system

GPAC: examples

Example (One variable, linear system)



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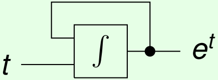

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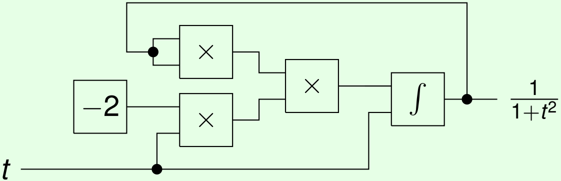

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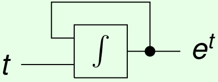

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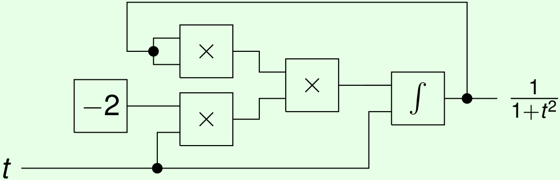

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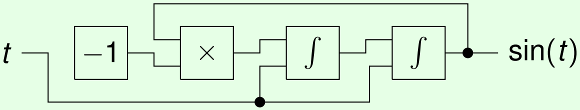

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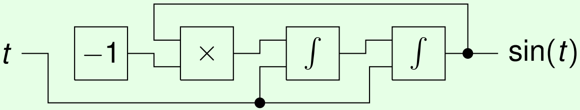

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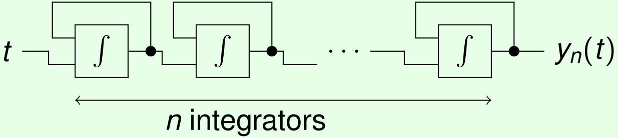

$$\begin{cases} y_1' = y_1 \\ y_2' = y_2 y_1' \\ \vdots \\ y_n' = y_n y_{n-1}' \end{cases}$$

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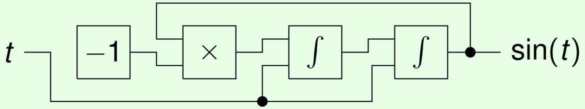

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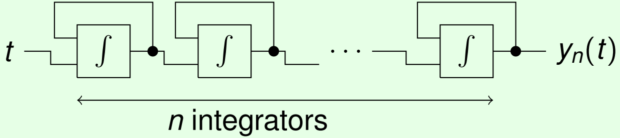

$$\begin{cases} y_1(t) = e^t \\ y_2(t) = e^{e^t} \\ \quad \dots \\ y_n(t) = e^{e^{\cdot^{\cdot^{\cdot^{t}}}}} \end{cases}$$

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Example (Non-computable real)

$$r = \sum_{n=0}^{\infty} d_n 2^{-n}$$

where

$$d_n = 1 \Leftrightarrow \text{the } n^{th} \text{ Turing Machine halts on input } n$$

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Example (Counter-Example)

$$f(x) = \lceil x \rceil$$

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Can we fix this ?

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Definition

y is **generated** by a GPAC iff it is a component of the solution $y = (y_1, \ldots, y_d)$ of the ordinary differential equation (ODE):

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- Simulate a Turing machine with a GPAC

$\square$

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First step: define a notion of complexity

Time Scaling

System	#1	#2
ODE	$\begin{cases} y'(t) = p(y(t)) \\ y(1) = y_0 \end{cases}$	$\begin{cases} z'(t) = u(t)p(z(t)) \\ u'(t) = u(t) \\ z(t_0) = y_0 \\ u(1) = 1 \end{cases}$

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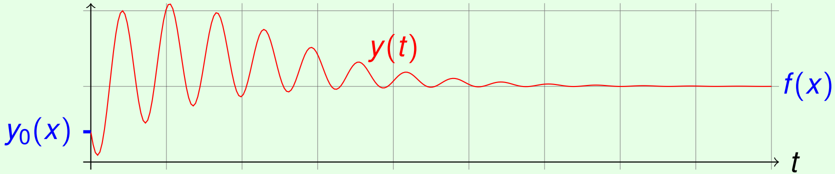
Remark

Same curve, different speed: $u(t) = e^t$ and $z(t) = y(e^t)$

Example



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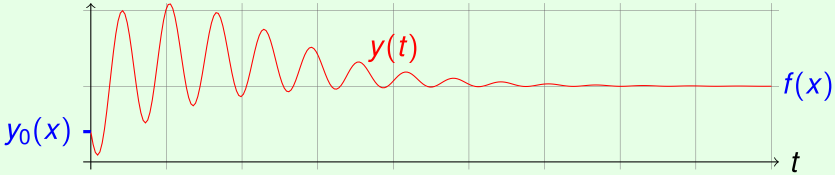

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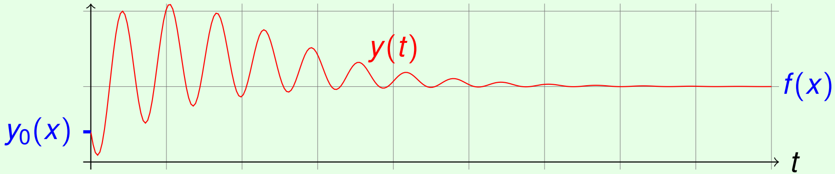

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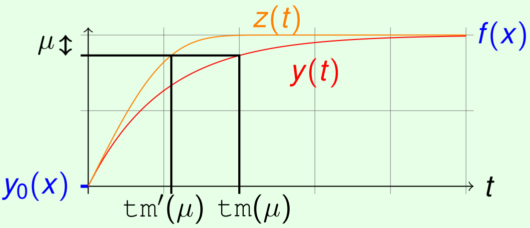

$$\|y_1(\mathtt{tm}(\mu)) - f(x)\| \leqslant \mu$$

Time Scaling

ODE	$y' = p(y)$	$\begin{cases} z' = up(z) \\ u' = u \end{cases}$
Computed Function	$f(x) = \lim_{t\to\infty} y_1(t) = \lim_{t\to\infty} z_1(t)$	
Time for precision μ	$\mathtt{tm}(\mu)$	$\mathtt{tm}'(\mu) = \log(\mathtt{tm}(\mu))$
Bounding box for ODE at time t	$\mathtt{sp}(t)$	$\mathtt{sp}'(t) = \max(\mathtt{sp}(e^t), e^t)$

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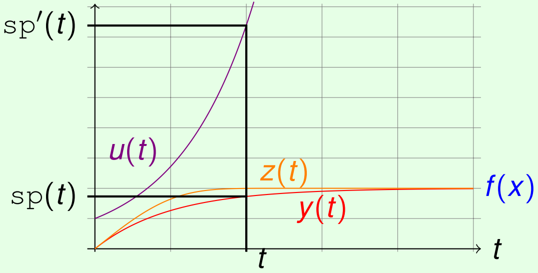

$$\mathtt{sp}(t) = \sup_{\xi\in[1,t]} \|y(\xi)\|$$

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Remark

- $\mathtt{tm}(\mu)$ and $\mathtt{sp}(t)$ depend on the convergence rate
- $\mathtt{sp}(\mathtt{tm}(\mu))$ seems not

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- Bounding Box at precision $\mu \Rightarrow$ Ok but geometric interpretation ?
- Length of the curve until precision $\mu \Rightarrow$ Much more intuitive

GPAC Computability (1)

Definition (GPAC-Computable Function)

$f \in GCOMP(\mathtt{sp}, \mathtt{tm})$ iff $\exists p, q$ polynomials, such that $\forall \alpha > 0$, $\forall \|x\| \leqslant \alpha$, $\exists y$ which satisfies:

- $\forall t \geqslant 0, y'(t) = p(y(t))$ and $y(0) = q(x)$
- $\forall \mu \geqslant 0, \forall t \geqslant \mathtt{tm}(\alpha, \mu), |f(x) - y_1(t)| \leqslant e^{-\mu}$
- $\forall t \geqslant 0, \|y(t)\| \leqslant \mathtt{sp}(\alpha, t)$

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Remark

- implies $f(x) = \lim_{t \to \infty} y_1(t)$
- can be extended to multi-dimensional functions
- can be defined over arbitrary input domain

GPAC Computability (2)

Definition (Polytime GPAC-Computable Function (Alternative))

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- $\forall t \geqslant 0, y'(t) = p(y(t))$ and $y(0) = q(x)$
- $\forall \mu \geqslant 0, \forall t \geqslant \ell^{-1}(\mathtt{len}(\alpha, \mu)), |f(x) - y_1(t)| \leqslant e^{-\mu}$
- $\ell(t)$ is the length of the curve y from 0 to t.

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GPAC Computability (2)

Definition (Polytime GPAC-Computable Function (Alternative))

$f \in GLEN(\mathtt{len})$ iff $\exists p, q$ polynomial such that $\forall \alpha > 0$, $\forall \|x\| \leqslant \alpha$, $\exists y$ which satisfies:

- $\forall t \geqslant 0, y'(t) = p(y(t))$ and $y(0) = q(x)$
- $\forall \mu \geqslant 0, \forall t \geqslant \ell^{-1}(\mathtt{len}(\alpha, \mu)), |f(x) - y_1(t)| \leqslant e^{-\mu}$
- $\ell(t)$ is the length of the curve y from 0 to t.

$$GPLEN = GLEN(\text{poly})$$

Remark

- implies $f(x) = \lim_{t\to\infty} y_1(t)$
- length of a curve: $\ell(t) = \int_{t_0}^{t} \|p(y(u))\| \, du$
- $\ell^{-1}(l) =$ time to travel a length l on the curve y

Computable Analysis = GPAC

Lemma (oversimplified)

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The polytime GPAC-computable functions (GP) are exactly the polytime computable functions of the Computable Analysis.

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- Any solution to a PIVP is polytime computable + exponential convergence

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- Any solution to a PIVP is polytime computable + exponential convergence
- Simulate a Turing machine with a GPAC

□

Proof sketch (1)

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$$\operatorname{poly}(\deg(p), L(t), \log \|y_0\|, \log \Sigma p, \mu)^d$$

where

$$L(t) = \int_0^t \Sigma p \max(1, \|y(u)\|)^{\deg(p)} du \approx \text{length of } y \text{ over } [0, t]$$

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Numerical analysis, for another talk ? □

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Example

- $\mathbb{R}[X] = [0, -1, 1, X; +, \times]$
- primitive recursive=$[0, S, \pi_i; \circ, REC]$
- recursive=$[0, S, \pi_i; \circ, REC, MU]$

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Proof.

Very technical $\square$

Proof sketch (4)

$f : \mathbb{R} \to \mathbb{R}$ polytime computable, $\mathcal{M}$ Turing Machine for f, $s_{\mathcal{M}}$ one step of $\mathcal{M}$

$$\begin{aligned} f(x) &= \lim_{\mu\to\infty} \mathcal{M}(x,\mu) \\ &= \lim_{\mu\to\infty} \lim_{n\to\infty} s_{\mathcal{M}}^{[n]}(x,\mu) \end{aligned}$$

and $s_{\mathcal{M}}$ can be built using composition and *GP*.

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  $f : \mathbb{N} \subseteq \mathbb{R} \to \{0,1\} \subseteq \mathbb{R}$
- Yes but there is more !

Definition (GPAC-Recognisable Language)

$\mathcal{L} \subseteq \mathbb{N}$ GPAC-recognisable if for any $x \in \mathbb{N}$, the solution y to

$$\begin{cases} y' = p(y) \\ y(t_0) = q(x) \end{cases}$$

where p,q are vectors of polynomials

satisfies for $t \geqslant t_1(x)$:

- if $x \in \mathcal{L}$ then $y_1(t) \geqslant 1$ (accept)
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What about complexity ?

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Remark (Why $\log(x)$?)

Classical complexity measure: length of word $\approx$ log of value

Definition (Non-deterministic Polytime GPAC-Recognisable Language)

$\mathcal{L} \subseteq \mathbb{N}$ non-deterministic poyltime GPAC-recognisable if for any $x \in \mathbb{N}$, the solution y to

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satisfies for $t \geqslant t_1(x)$:

- if $x \in \mathcal{L}$ then $y_1(t) \geqslant 1$ for at least one digital controller u
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Digital Controller $\approx u : \mathbb{R} \to \{0, 1\}$

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Theorem

The class of non-deterministic polytime GPAC-recognisable languages is exactly *NP*.

Conclusion

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- Equivalence with Computable Analysis for polynomial time

Future Work

- Notion of reduction ?

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- Space complexity ?

Questions ?

- Do you have any questions ?