Constraint information inequalities: geometric, algorithmic and combinatorial views

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a joint work with
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Outline

1. Shannon entropy: basic definitions
2. "Standard" information inequalities
3. "Conditional" information inequalities
4. Conditional inequalities: geometric view
5. Information inequalities for Kolmogorov complexity
6. Towards combinatorial interpretation

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distribution: $\begin{array}{c|c|c|c} s_1 & s_2 & \dots & s_k \\ \hline p_1 & p_2 & \dots & p_k \end{array}$, $p_i \geq 0$, $\sum p_i = 1$

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Measure of *uncertainty* in α : $0 \leq H(\alpha) \leq \log k$

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[**Shannon type** ineq] == [combinations of basic ineq]:

example 1: $H(a) \leq H(a\,|\,b) + H(a\,|\,c) + I(b:c)$

example 2: $2H(a,b,c) \leq H(a,b) + H(a,c) + H(b,c)$

Linear information inequalities

General form: A linear information inequality is a combination of reals $\{\lambda_{i_1,\dots,i_k}\}$ such that

$$\sum \lambda_{i_1,\dots,i_k} H(a_{i_1},\dots,a_{i_k}) \geq 0$$

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Applications:

- multi-source network coding
- secret sharing
- **combinatorial interpretations**
- group theoretical interpretation
- **Kolmogorov complexity**
- ...

Once again, Shannon type information inequalities :

- subadditivity,
  $H(A \cup B) \leq H(A) + H(B)$
  [in other notation $I(A : B) \geq 0$]
- submodularity,
  $H(A \cup B \cup C) + H(C) \leq H(A \cup C) + H(B \cup C)$
  [in other notation $I(A : B|C) \geq 0$]
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Th [Z. Zhang, R.W. Yeung 1998] There exists a non-Shannon type information inequality:

$$\begin{aligned} I(c : d) \ \leq \ & 2I(c : d \,|\, a) + I(c : d \,|\, b) + I(a : b) \\ & + I(a : c \,|\, d) + I(a : d \,|\, c) \end{aligned}$$

Theorem [Z. Zhang, R.W. Yeung 1997] There exists a *conditional* non Shannon type inequality:

$$I(x:y) = I(x:y\,|\,a) = 0$$

$$\Downarrow$$

$$I(a:b) \le I(a:b\,|\,x) + I(a:b\,|\,y)$$

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(a) **Trivial, Shannon-type**:

if $I(x : y) = 0$ then $H(a) \leq H(a \,|\, x) + H(a \,|\, y)$

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if $I(x:y)=0$ then $H(a) \le H(a\,|\,x) + H(a\,|\,y)$

this is true since $H(a) \le H(a\,|\,x) + H(a\,|\,y) + I(x:y)$
[Shannon-type unconditional inequalitiy]

Conditional information inequalities

(b) **Trivial, non Shannon-type**:

if $I(a:b\,|\,z) = I(a:z\,|\,b) = I(b:z\,|\,a) = 0$ then

$$I(a:b) \le I(a:b\,|\,x) + I(a:b\,|\,y) + I(x:y)$$

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(b) **Trivial, non Shannon-type**:

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this is true since

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[non Shannon-type unconditional inequalitiy]

Conditional information inequalities

(c) **Non trivial, non Shannon-type**:

- **Zhang,Yeung 97:** if $I(x : y) = I(x : y \mid a) = 0$ then
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- **Tarik Kaced and A.R. 2011:** if $H(a\,|\,x,y) = I(x:y\,|\,a) = 0$ then
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Theorem. These three statements are *essentially* conditional inequalities.

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That is, for all λ_1, λ_2 the inequality

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does not hold.

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We cannot reduce it to an unconditional inequality!

More precisely, for all λ_1, λ_2 there exist (a, b, c, d) such that

$$I(a:b) \not\leq I(a:b|x) + I(a:b|y) + I(x:y) + \lambda_1 H(a|x,y) + \lambda_2 I(x:y\,|\,a)$$

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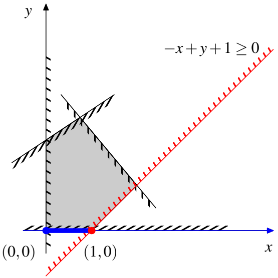
A geometric view on conditional inequalities:



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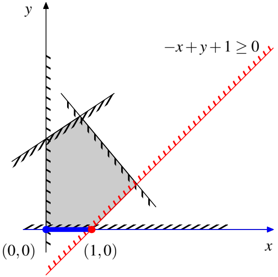

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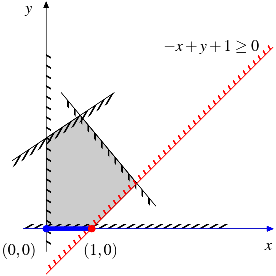

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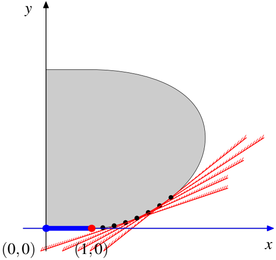

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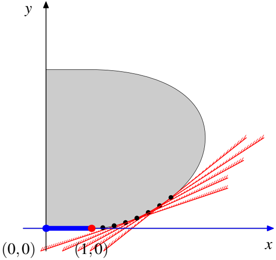

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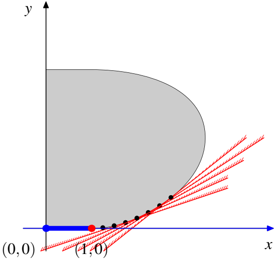

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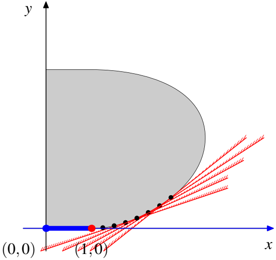

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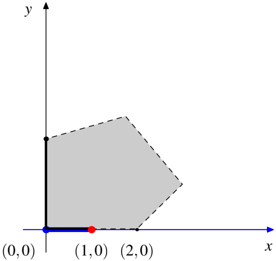

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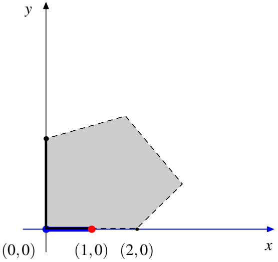

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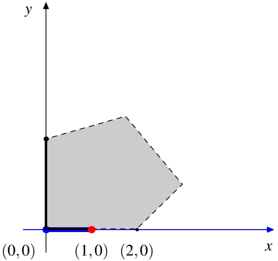

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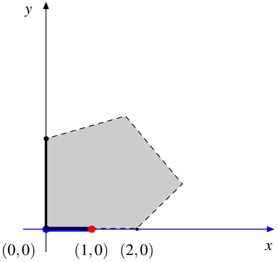

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the general scheme: $\lambda_1 H(a_1) + \lambda_2 H(a_2) + \ldots + \lambda_{12} H(a_1, a_2) + \ldots \geq 0$

is equivalent to

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$I(x : a|b) \le \sqrt{N}$ & $I(x : b|a) \le \sqrt{N} \Rightarrow I(a : b) \le I(a : b \mid x) + I(a : b \mid y) + I(x : y) + O(N^{3/4})$

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- Essentially conditional inequalities [Zhang–Yeung'97] and [Kaced–R.'11] are **not** for Kolmogorov complexity

Inequalities for Kolmogorov complexity:

- Exactly the same classes of unconditional linear inequalities hold for Shannon's entropy and for Kolmogorov complexity.

  e.g. $H(a_1, a_2) \leq H(a_1) + H(a_2)$ vs $C(a_1, a_2) \leq C(a_1) + C(a_2) + O(\log N)$

  the general scheme: $\lambda_1 H(a_1) + \lambda_2 H(a_2) + \ldots + \lambda_{12} H(a_1, a_2) + \ldots \geq 0$

  is equivalent to

  $$\lambda_1 C(a_1) + \lambda_2 C(a_2) + \ldots + \lambda_{12} C(a_1, a_2) + \ldots + O(\log N) \geq 0$$

- Essentially conditional inequality [Matúš'99] is valid for Kolmogorov complexity (in *some* sense).

  $I(x : a|b) \leq \sqrt{N}$ & $I(x : b|a) \leq \sqrt{N} \Rightarrow I(a : b) \leq I(a : b \mid x) + I(a : b \mid y) + I(x : y) + O(N^{3/4})$

- Essentially conditional inequalities [Zhang–Yeung'97] and [Kaced–R.'11] are **not** for Kolmogorov complexity (in any reasonable sense)

Outline

Once again,

Theorem: $H(a|x, y) = I(x : y|a) = 0 \Rightarrow I(a : b) \leq I(a : b \mid x) + I(a : b \mid y) + I(x : y)$

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Theorem: $H(a|x,y) = I(x:y|a) = 0 \Rightarrow I(a:b) \leq I(a:b\,|\,x) + I(a:b\,|\,y) + I(x:y)$

Why is it valid?

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Why is it valid? And what does it mean?

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Why is it valid? And what does it mean?

We relax the constraint and make the inequality stronger:

(1) a is a function of (x, y)

(2) $\Pr[X_i\,|\,A_k] > 0$ & $\Pr[Y_j\,|\,A_k] > 0 \Rightarrow \Pr[X_i, Y_j\,|\,A_k] > 0$

Theorem:

$$(1) + (2) \implies H(a\,|\,x,b) + H(a\,|\,y,b) \leq H(a\,|\,b)$$

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$$\Downarrow$$

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- $G =$ bi-partite graph

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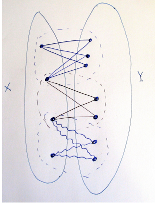

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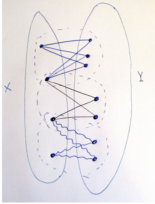

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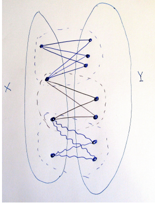

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Our conditions:

(1) a is uniquely defined by x and y

(2) edges of each color make a clique

Th. (1) + (2) $\Rightarrow H(a \mid x, b) + H(a \mid y, b) \leq H(a \mid b)$

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Th. (1) + (2) $\Rightarrow H(a \mid x, b) + H(a \mid y, b) \leq H(a \mid b)$

Proof:

- keep the distribution of (a, b),

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Th. (1) + (2) $\Rightarrow H(a \mid x, b) + H(a \mid y, b) \leq H(a \mid b)$

Proof:

- keep the distribution of (a, b), take x and y independently given (a, b)

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- keep the distribution of (a, b), take x and y independently given (a, b)
- trivial: $H(a, b, x', y') = H(a, b) + H(x \mid a, b) + H(y \mid a, b)$

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- result: $H(a,b) + H(x\,|\,a,b) + H(y\,|\,a,b) \le H(b) + H(x\,|\,b) + H(y\,|\,b)$
- which implies $H(a\,|\,x,b) + H(a\,|\,y,b) \le H(a\,|\,b)$

So what?

So what? Why do we need all these inequalities?

This is all about (bi-)clique covering!

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Let G be bi-partite graph with colored edges.

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Recipe:

- take a distribution (a, x, y) [e.g., a uniform distribution on edges]
- add a suitable b [don't ask me how to invent this b]
- observe $H(a \mid x, b) + H(a \mid y, b) \not\leq H(a \mid b)$

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- then [bi-clique covering number] $\ge 2^{(H(a\,|\,x,b)+H(a\,|\,y,b)-H(a\,|\,b))/2}$

Open problems:

1. **[geometry]** The form of the cone of "entropic" points: infinitely many flat facets? or a curved surface?
2. **[complexity]** Another conditional inequality by F. Matúš: is it valid for Kolmogorov complexity?
3. **[combinatorics/complexity]** Lower bounds for clique covering: find examples where this technique is more effective then the conventional arguments (applications to communication complexity).